ON REPRESENTATIONS OF FINITE GROUPS WITH SPLIT BN-PAIRS

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A thesis submitted for the degree of Doctor of Philosophy at the University of Warwick, 1987.

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ACKNOWLEDGEMENTS

I owe a great debt to many people, without whom this work would not have been possible.

To my supervisor, Professor J.A. Green, who initiated my interest in this subject and for his patience and guidance during my research time at Warwick University.

To Professor R. Carter who has been always ready to give advices and helps whenever I needed them.

To my parents, brothers and sisters who have been a great source of inspiration and encouragement during my education time.

To Umm - ALQura University, Makkah, Saudi Arabia, for their generous financial support.

Finally to Peta McAllister for doing such an elegant typing job.

Some Standard Notations

$\|X\|$	the cardinality of a set X
$X \subseteq T$	X is a subset of T
$T \backslash X$	the complement of X in T

If G is a group

$H \leq G$	H is a subgroup of G
$H \leq G$	H is a normal subgroup of G
$<H_1,\ldots,H_t>$	the subgroup of G generated by $H_1,\ldots,H_t \subseteq G$

If k is a field and M is a kG-module affording the character ρ of G

$M\|_H$	the restriction of M to H
$\rho\|_H$	the restriction of ρ to H

If L is a kH-module $(H \leq G)$

$L^G = \mathrm{Ind}_H^G L$	the kG-module induced from L
char k	the characteristic of the field k
$\dim_k M$	the dimension of a k-vector space M
$\underline{r}\Lambda$	the Jacobson radical of an algebra Λ

In chapter 2 and 3, k is an arbitrary field unless otherwise is stated.

Throughout part II, (K,R,F) is a p-modular system, where $p > 0$ is the characteristic of a finite group $G = (G,B,N,\underline{R},U)$ with a split BN-pair and k is any field such that char $k \nmid |H|$ and k is a splitting field for $H = B \cap N$.

INTRODUCTION.

Let $G = (G,B,N,\underline{R},U)$ be a finite group with a split BN-pair of characteristic p and rank ℓ . Let $(W,\underline{R})$ be the Coxeter system of G . Let (K,R,F) be a p-modular coefficient system such that K and F are splitting fields for G and all its subgroup. The permutation FG-module $FY = FG[U]$, where $[U] = \sum_{u \in U} u$, plays an important role in the modular representations of G , since it contains every simple FG-module as a composition factor.

This thesis deals with various subjects concerning the FG-module FY . It is divided into two parts (although the division is strictly not necessary). Part I consists of the first three chapters. The main purpose of Part I is to investigate some kG-modules (k is a field) arising from some functors defined by Green in [G1].

Chapter 1 contains the basic definitions and some structure theorems of finite groups with BN-pairs.

Chapter 2 (§2.1) contains an outline of Auslander theory on the category Cov Λ whose objects are functors defined on the category mod Λ of all finite dimensional left Λ-modules, where Λ is a finite dimensional algebra over a field k .

For $V, X \in \text{mod } \Lambda$, Green [G1] defined three functors

$$\ell_{V,X} \text{ , } r_{V,X} \text{ , } q_{V,X} : \text{mod } E(V) \to \text{mod } E(X) \text{ ,}$$

which connect the representations of the two algebras $E(V) = \text{End}_\Lambda(V)$ and $E(X) = \text{End}_\Lambda(X)$. The functor $q_{V,X}$ sends every simple left $E(V)$-module to a simple left $E(X)$-module $q_{V,X}(M)$, if $q_{V,X}(M) \neq 0$. §2.2 contains the description of these three functors together with some adaptation to the theory of Auslander on the classification of the simple objects in $\mathrm{Cov}\,\Lambda$. In this section, we take $\Lambda = kG$, the group algebra of a finite group over a field k. We consider the $E(X)$-modules $\ell_{V,X}(M)$, $r_{V,X}(M)$ and $q_{V,X}(M)$ in the case where $M = k_\psi$, a 1-dimensional left $E(V)$-module affording a 1-dimensional character ψ of $E(V)$.

In §2.3, we take $V = kG[H]$, $H \leq G$ and $X = {}_{kG}kG$, the regular left kG-module. We study the right kG-module $r_{kG[H],kG}(k_\psi)$, when ψ is a 1-dimensional character of $\mathrm{End}_{kG}(kG[H])$.

In chapter 3, we take $G = (G,B,N,\underline{R})$ to be a finite group with a BN-pair whose Coxeter system is $(W,\underline{R})$. The k-algebra $E(kG[B])$ has a k-basis $\{A_w ; w \in W\}$ indexed by the elements of W. We study the right kG-module $r_{kG[B],kG}(k_\psi)$, where $\psi : E(kG[B]) \to k^\times$ is the multiplicative character of $E(kG[B])$ given by $\psi(A_w) := (-1)^{\ell(w)}$ $(w \in W)$ where ℓ is the length function on the elements of W. We show that $r_{kG[B],kG}(k_\psi)$ is isomorphic to the homology module $H_{\ell-1}(\Delta)$ (Theorem 3.1.14), where Δ is the simplicial complex of G defined by Tits [JT] and $\ell = |\underline{R}|$. Consequently, we are able to recover some well-known results about the Steinberg representation of finite groups with BN-pairs (§3.2). In particular, we give an easy proof of the Tits-Solomon theorem on $H_{\ell-1}(\Delta)$ (their proof involves a less obvious geometrical argument).

Part II.

Chapter 4 contains an introduction to the modular representations of finite groups with split BN-pairs. The structure of the R-order $E(Y) = \mathrm{End}_{RG}(RG[U])$ was discussed in §4.1. In §4.2, we outline the theory of Curtis-Richen on the classification of the simple FG-modules, where $G = (G,B,N,\underline{R},U)$ is a finite group with a split BN-pair.

The RG-lattice $Y = RG[U]$ has a decomposition $Y = \sum^{\oplus}_{\chi \in \tilde{H}} Y_\chi$, where $H = B \cap N$, $\tilde{H} = \mathrm{Hom}(H,K^\times)$ and Y_χ $(\chi \in \tilde{H})$ is the weight subspace of Y of weight χ (Prop. 4.1.10).

In chapter 5, an R-order S_χ is introduced, for every $\chi \in \tilde{H}$. The R-orders $\{S_\chi, \chi \in \tilde{H}\}$ contains all the information about the R-order $E(Y)$. We give an R-basis for S_χ $(\chi \in \tilde{H})$ (Prop. 5.0.7) and in §5.1, we study the F-algebra $FS_\chi = F \otimes_R S_\chi$, where $\chi \in \tilde{H}$ is regular. We determine all the simple right FS_χ-modules and its Cartan matrix which turns out to be singular in general.

In chapter 6, we give a formula for calculating the characters of the RG-summands of $Y = RG[U]$, using the results of N. Tinberg [NT2] introduced in §4.2.

In chapter 7, we discuss the decomposition matrix D_χ of the Hecke algebra $\mathrm{End}_{KG}(KY_\chi)$ $(\chi \in \tilde{H})$, using a recent theorem, due to Green [G2], which gives an interpretation of the decomposition numbers as multiplicities of ordinary characters of G.

In §7.2, we show that the problem of calculating D_χ $(\chi \in \tilde{H})$ can be reduced to the case of the Levi subgroup of some parabolic subgroup G_J of G. In the end of §7.2, we assume that $G = G(q)$ is a member of a system of finite groups with BN-pairs. We appeal to some results in [CIK] and show that the decomposition numbers of the Hecke algebra $E(KG[B])$ can be interpreted as multiplicities of ordinary characters of the Weyl group W of G (Prop. 7.2.31).

In §7.3, a notion of θ-contravariant forms on Y_χ $(\chi \in \tilde{H})$ is introduced, where $\theta : G \to G$ is an anti-automorphism of G satisfying certain axioms. We show that the set $\{Y_\chi \,;\, \chi \in \tilde{H}\}$ is closed under the θ-duality and that the simple FG-modules are self θ-dual. Consequently, we are able to show that the decomposition matrices D_χ and $D_{w_0\chi}$ of the Hecke algebras $E(KY_\chi)$ and $E(KY_{w_0\chi})$, respectively, are identical, where w_0 is the unique element of W of maximal length.

In §7.4, we consider the direct product of two finite groups with split BN-pairs. We prove some results which are relevant to the subject of chapter 7.

§7.5 contains an application of the previous results to the case of the general linear group.

CHAPTER 1. BN-pairs

§1.1 Groups with BN-pairs

Definition 1.1.1 (Tits [JT]): A group G is said to have a BN-pair if there exist subgroups B and N of G such that,

(i) $G = \langle B,N \rangle$, $B \cap N \triangleleft N$.

(ii) The group $W = N/B\cap N$ is finite and is generated by a set of involutions $\underline{R} = \{w_1,\ldots,w_\ell\}$.

(iii) For all $w_i \in \underline{R}$ and $w \in W$, $w_i\, B\, w \subset BwB \cup B\, w_i w\, B$, and

(iv) For all $w_i \in \underline{R}$, $w_i\, B\, w_i \neq B$.

The group $W = N/B\cap N$ is called the Weyl group of the BN-pair, and ℓ is the rank. Write $H = B\cap N$, and let $t : N \to W$ be the natural map. The notation wB, Bw, $w \in W$ means nB, Bn, respectively, for any $n \in N$ such that $t(n) = w$; note that they are well-defined since $H \triangleleft N$, $H \leq B$.

The following theorem relates the (B,B)-double cosets in a group G with BN-pair to the elements of the Weyl group W.

Bruhat theorem 1.1.2 ([NB], Th.1, p.25) Let G be a group with a BN-pair with Weyl group W then:

(i) $G = \dot{\bigcup\limits_{w\in W}} BwB$,

(ii) $BwB = Bw'B$ implies $w = w'$, for $w,w' \in W$. □

Matsumoto [HM] showed that the set $\underline{R}$ generates W as a coxeter group; thus W has a presentation

$$W = \langle w_i \in \underline{R} \ / \ w_i^2 = 1 \ , \ (w_i w_j)^{n_{ij}} = 1 \rangle \quad ,$$

where n_{ij} is the order of $w_i w_j$ $(i \neq j)$. Therefore W is isomorphic to a finite group generated by reflections in ℓ-dimensional euclidean space ([NB] pp.25, 91). To such finite reflection group one can associate a set of roots ([RC], 2.2); we will denote the set of roots by Φ.

Φ has a subset $\Pi = \{\alpha_1,\ldots,\alpha_\ell\}$, whose elements are called the fundamental roots, such that every $w_i \in W$ can be identified with the reflection in the hyperplane orthogonal to α_i, and such that every $\alpha \in \Phi$ is a linear combination $\sum_{i=1}^{\ell} \lambda_i \alpha_i$ where either each λ_i is non-negative or each λ_i is non-positive.

Let
$$\Phi^+ = \{\alpha \in \Phi : \alpha = \Sigma \lambda_i \alpha_i \ , \ \lambda_i \geq 0\} \quad ,$$
$$\Phi^- = \{\alpha \in \Phi : \alpha = \Sigma \lambda_i \alpha_i \ , \ \lambda_i \leq 0\} \quad ;$$

Φ^+, Φ^- are called the sets of positive and negative roots respectively. Every root is the image of some fundamental root under some element of W.

If $w \in W$, let $\ell(w)$ be the minimal number of terms in an expression of w as a product of generators w_i. $w = w_{i_1} \ldots w_{i_s}$ is called reduced expression for w, if $s = \ell(w)$. W has a unique element w_0 of maximal length. We have $\ell(w_0) = |\Phi^+|$, $w_0(\Phi^+) \subset \Phi^-$, and $w_0^2 = 1$.

For every subset $J \subseteq \underline{R}$, let $W_J = \langle J \rangle$. The elements of J generate W_J as a coxeter group. The subgroups W_J ; $J \subseteq \underline{R}$, are called the parabolic subgroups of W. If we let V_J be the set of linear combinations of the set $\{\alpha_i \ / \ w_i \in J\}$, then W_J acts on V_J as a euclidean reflection group. Let $\Phi_J = \Phi \cap V_J$. Φ_J is the root system of W_J . The set $\Pi_J = \Pi \cap V_J = \{\alpha_i \ / \ w_i \in J\}$ forms the set of fundamental roots in Φ_J . Let $G_J = BW_JB$.

Definition 1.1.3 A Borel subgroup of G is a subgroup conjugate to B . A parabolic subgroup of G is a subgroup conjugate to G_J for some $J \subseteq \underline{R}$.

Theorem 1.1.4 (Tits, [JT]):

(1) The subgroups G_J ; $J \subseteq \underline{R}$, are the only subgroups of G containing B .

(2) Two different parabolic subgroups which contain a common Borel subgroup are not conjugate in G .

(3) The normalizer of a parabolic subgroup is itself.

$\square$

Lemma 1.1.5 If G has a BN-pair $(G,B,N,\underline{R})$ then, for all $J \subseteq \underline{R}$, G_J has a BN-pair (G_J,B,N_J,J), where N_J is the inverse image of W_J under t .

$\square$

Theorem 1.1.6 (Curtis, [C1]): Let G be a finite group with a BN-pair $(G,B,N,\underline{R})$, and let W be the Weyl group of G . Then the map $G_J \to W_J$ $(J \subseteq \underline{R})$ gives a bijection between the family of parabolic subgroups G_J of G , which contain B , and the family of parabolic subgroups W_J of W . Furthermore there is a well defined bijection $W_J \, w \, W_K \to G_J \, w \, G_K$ $(w \in W$ and $J,K \subseteq \underline{R})$ between the set of (W_J,W_K)-cosets of W , and the set of (G_J,G_K)-cosets of G .

§1.2 Groups with split BN-pairs

Definition 1.2.1 A group G is said to have a split BN-pair of rank ℓ and characteristic p, for some prime $p > 0$, if:

(i) G has a BN-pair $(G,B,N,\underline{R})$ of rank ℓ.

(2) $B = UH$ where U is normal p-subgroup of B, and H is p'-subgroup of B.

(3) $H = \bigcap_{n \in N} B^n$.

The axiom (3) is called "the saturation axiom". We write $(G,B,N,\underline{R},U)$ for the split BN-pair of G.

For each $w \in W$ we choose $(w) \in N$ such that $t((w)) = w$ (i.e. $(w)H = w \in W$).

Definition 1.2.2 For every $w \in W$ define:

$$U_w^+ = U \cap U^w , \quad U_w^- = U \cap U^{w_o w}$$

$$U_i = U_{w_i}^- = U \cap U^{w_i} , \quad U_{-i} = {}^{w_i}U_i , \quad U^- = U^{w_o} .$$

Since H normalizes U, these definitions are independent of the choice of the coset representatives (w).

The proof of the following consequences of the axioms of the split BN-pair can be found either in [C2] or [FR].

Theorem 1.2.3 ([FR], 3.4): For $w \in W$,

$$U = U_w^- U_w^+ = U_w^+ U_w^- \text{ and } U_w^- \cap U_w^+ = \{1\} . \qquad \square$$

Definition ([CL], p.351). For $w_i \in \underline{R}$, let $H_i = \langle U_i, U_{-i}\rangle \cap H$.

Lemma 1.2.4 (See [FR])

(1) The coset representative (w_i) can be chosen in $\langle U_i, U_{-i}\rangle$.

(2) $\langle U_i, U_{-i}\rangle = U_iH_i \dot{\cup} U_iH_i(w_i)U_i$.

(3) The subgroups U_iH_i, $\langle H_i,(w_i)\rangle$ form a split BN-pair in $\langle U_i, U_{-i}\rangle$.

(4) If $x \in U_i^*$, then $(w_i)^{-1}x(w_i) \in U_iH_i(w_i)U_i$.

Example 1.2.5 $G = GL(n,q)$, the group of non-singular $n\times n$ matrices with coefficients in $GF(q)$ where q is a power of p, for some prime $p > 0$. Let B be the subgroup of upper triangular matrices in G, and let N be the subgroup of monomial matrices in G, i.e. $N = \{(a_{ij}) \in G$, (a_{ij}) has exactly one non-zero entry in each row and each column$\}$. Then we have, $H = B \cap N$ = subgroup of diagonal matrices. H is normal p'-subgroup in N, and $N/H = W \cong S_n$, the symmetric group on n letters. The group S_n can be generated by transpositions, $\underline{R} = \{w_1 = (12), w_2 = (23), \ldots, w_{n-1} = (n-1,n)\}$.

Let U be the subgroup of upper unitriangular matrices in G, U is normal p-subgroup of B. We also have $B = UH$, $U \cap H = \{1\}$. It is not difficult to check that the axioms for the BN-pair are satisfied, and so $(G,B,N,\underline{R},U)$ is a split BN-pair for G (see [CRIII], p.580).

The following theorem is a consequence of 1.2.3 (see [RC], Theorem 2.5.14).

Theorem 1.2.6 Each element of a group G with split BN-pair

$(G,B,N,\underline{R},U)$ is uniquely expressible in the form $uh(w)u'$ where $u \in U$, $h \in H$, $w \in W$, and $u' \in U_w^-$.

□

Definition 1.2.7 For every $w \in W$, let

$$\Pi_w^- = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\} \quad ,$$

$$\Pi_w^+ = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+\} \quad .$$

In the next proposition, a collection of subgroups $\{U_\alpha ; \alpha \in \Phi\}$ of G are defined on which the Weyl group W of G acts in the same way that W acts on Φ . For every $\alpha_i \in \Pi$, let $U_{\alpha_i} := U_i$.

Proposition 1.2.8 (See [CRIII], Proposition 69.2): Let $G = (G,B,N,\underline{R},U)$ be a finite group with a split BN-pair of rank ℓ and characteristic p . Then

(i) There exists a bijection $\alpha \to U_\alpha$ from the root system Φ to the set of conjugates $\{{}^nU_i : n \in N , \alpha_i \in \Pi\}$, which extends the map $\alpha_i \to U_i$, $\alpha_i \in \Pi$. For all $w \in W$ and $\alpha \in \Phi$, we have

$$^{(w)}U_\alpha = U_{w(\alpha)} \quad .$$

(ii) For each $w \in W$ we have

$$U_w^- = \prod_{\alpha \in \Pi_w^-} U_\alpha \quad ,$$

with uniqueness of expression. A similar statement holds for U_w^+ and Π_w^+ . In particular, $U_w^\pm = \langle U_\alpha : \alpha \in \Pi_w^\pm \rangle$.

CHAPTER 2. The Functor Category

Throughout Chapter 2, k is a field, and Λ is any finite dimensional k-algebra.

§2.0 Definitions and preliminaries

Let Λ be a finite dimensional k-algebra. By $\mathrm{mod}\,\Lambda$ we mean the category of all finitely generated left Λ-modules and we denote by $\mathrm{Cov}\,\Lambda$ [$\mathrm{Fun}\,\Lambda$] the category whose objects are all covariant [contravariant] k-linear functors $F : \mathrm{mod}\,\Lambda \to \mathrm{Mod}\,k$, where $\mathrm{Mod}\,k$ denotes the category of all vector spaces over k. If $F_1, F_2 \in \mathrm{Cov}\,\Lambda$ [$\mathrm{Fun}\,\Lambda$], the morphisms $\alpha : F_1 \to F_2$ are natural transformations from F_1 to F_2 ([R] p.43). If $V, X \in \mathrm{mod}\,\Lambda$, we let $(X,V)_\Lambda = \mathrm{Hom}_\Lambda(X,V)$ and $E(V) = \mathrm{End}_\Lambda(V) = (V,V)_\Lambda$. For every Λ-module V, the functor $(V, \ )$ [$(\ ,V)$], which sends X mod Λ to the k-space $(V,X)_\Lambda$ [$(X,V)_\Lambda$], is an object of $\mathrm{Cov}\,\Lambda$ [$\mathrm{Fun}\,\Lambda$]. If $F',F \in \mathrm{Cov}\,\Lambda$ we say that F' is a <u>subfunctor</u> of F (written $F' \leq F$) if for every $X \in \mathrm{mod}\,\Lambda$, $F'(X)$ is a k-subspace of $F(X)$ and the inclusion map $i_X : F'(X) \to F(X)$ is natural in X, that is if $Y \in \mathrm{mod}\,\Lambda$ and $f \in (X,Y)_\Lambda$ then the following diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ i_X \uparrow & & \uparrow i_Y \\ F'(X) & \xrightarrow[F'(f)]{} & F'(Y) \end{array}$$

commutes.

A functor $F \in \mathrm{Cov}\,\Lambda$ is said to be <u>simple</u> if it is non-zero and has no proper subfunctor.

If $F' \leq F \in \mathrm{Cov}\,\Lambda$ we define the <u>quotient functor</u> $F/F' \in \mathrm{Cov}\,\Lambda$ to be the functor given by

$$(F/F')(X) := F(X)/F'(X) \quad \text{for all} \quad X \in \mathrm{mod}\,\Lambda \;,$$

and F/F' takes any morphism $f \in (X,Y)_\Lambda$, $X,Y \in \mathrm{mod}\,\Lambda$, to the k-map

$$(F/F')(f) : F(X)/F'(X) \to F(Y)/F'(Y)$$

induced by $F(f) : F(X) \to F(Y)$. If $X_1, X_2 \in \mathrm{mod}\,\Lambda$, and X_1 is Λ-submodule of X_2 , we write $X_1 \underset{\Lambda}{\leq} X_2$.

§2.1 The Simple Objects of Cov Λ : ([MA], [G1])

In [MA], Auslander studied the category Fun Λ (he named it Mod mod Λ) and characterized the simple objects of this category. In this section we outline the theory of Auslander, using the category Cov Λ instead of Fun Λ .

Given $F \in \mathrm{Cov}\,\Lambda$ and $V \in \mathrm{mod}\,\Lambda$, the k-space $F(V)$ has a structure of a left $E(V)$-module by setting:

$$h.x := F(h).x \quad \text{for all} \quad h \in E(V) \;,\; x \in F(V) \;.$$

The "<u>evaluation functor</u>" $e_V : \mathrm{Cov}\,\Lambda \to \mathrm{Mod}\,E(V)$, for every $V \in \mathrm{mod}\,\Lambda$ is given by:

$$e_V(F) := F(V) \text{ and } e_V(\alpha) := \alpha_V : F_1(V) \to F_2(V) \;,$$

for any object F and morphism $\alpha : F_1 \to F_2$ in Cov Λ .

The functor e_V induces a map:

$$e_V : \{\text{all sub functors } F' \leq F\} \to \{\text{all } E(V)\text{-submodules of } F(V)\} .$$

Definition ([G1], 3.1): If $M \underset{E(V)}{\leq} F(V)$, and $X \in \mathrm{mod}\,\Lambda$ define:

$$b_V(M)(X) := \{x \in F(X) \mid F(g)(x) \in M , \ \forall\, g \in (X,V)\} .$$

$b_V(M)$ defines a subfunctor of F, and b_V induces a map:

$$b_V : \{\text{all } M \underset{E(V)}{\leq} F(V)\} \to \{\text{all } F' \leq F\} .$$

Proposition 2.1.1 ([A], p.281, see also [G1], §9): Given F and V as above:

(i) $e_V b_V(M) = M$ for all $M \underset{E(V)}{\leq} F(V)$.

(ii) $F' \leq b_V e_V(F')$ for all $F' \leq F$.

(iii) $b_V(F(V)) = F$.

(iv) If M_1 and M_2 are $E(V)$-submodules of $F(V)$, then $M_1 \leq M_2$ implies $b_V(M_1) \leq b_V(M_2)$.

(v) If $F' \underset{\max}{<} F$ then either $F'(V) = F(V)$ or $F'(V) \underset{\substack{E(V)\\ \max}}{<} F(V)$ and $b_V e_V(F') = F'$.

Here $F' \underset{\max}{<} F$, $F'(V) \underset{\substack{E(V)\\ \max}}{<} F(V)$ means maximal subfunctor, maximal $E(V)$-submodules, respectively.

Proof

(i) Let $F' = b_V(M)$ and consider $e_V(F')$.

$$e_V(F') = F'(V)$$

$$= \{x \in F(V) \mid F(g)(x) \in M \ , \ \forall g \in E(V)\} \ .$$

By putting $g = 1_{E(V)}$, we have

$$x \in e_V(F') \text{ implies } x \in M \ , \text{ hence } e_V(F') \subseteq M \ .$$

Conversely suppose $x \in M$, then for every $g \in E(V)$, $F(g)(x) = g.x \in M$, since M is $E(V)$-submodule of $F(V)$. Therefore $x \in F'(V)$ $(= e_V(F'))$ and so

$$e_V b_V(M) = e_V(F') = F'(V) = M \ .$$

(ii) Given $X \in \text{mod } \Lambda$, we show that $F'(X) \leq (b_V e_V(F'))(X)$. By the definition we have $b_V e_V(F') = b_V(F'(V))$ and

$$b_V(F'(V))(X) = \{x \in F(X) \mid F(g)(x) \in F'(V) \text{ for all } g \in (X,V)\} \ .$$

But since $F' \leq F$, it follows that for every $g \in (X,V)$ we have a commutative diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(V) \\ \uparrow i_X & & \uparrow i_V \\ F'(X) & \xrightarrow{F'(g)} & F'(V) \end{array} \ .$$

Hence,

$$x \in F(X) \Rightarrow F(g)(x) \in F'(V) \text{ for all } g \in (X,V) \ ;$$

i.e. $\qquad x \in b_V(F'(V))(X) = b_V e_V(F')(X)$.

Therefore $F'(X) \leq (b_V e_V(F'))(X)$ for all $X \in \text{mod } \Lambda$.

To complete the proof we need to show that the inclusion map

$i_X : F'(X) \to (b_V e_V(F'))(X)$ is natural in X . So let $\theta : X \to V$ be a Λ-map $(X, V \in \text{mod}\ \Lambda)$. Consider the following diagram:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\theta)"] \arrow[dr, phantom, "1"] & F(V) \\
b_V e_V(F')(X) \arrow[u] \arrow[r, "b_V e_V(F')(\theta)"] \arrow[dr, phantom, "2"] & b_V e_V(F')(V) \arrow[u] \\
F'(X) \arrow[u, "i_X"] \arrow[r, "F'(\theta)"] & F'(V) \arrow[u, "i_V"']
\end{tikzcd}$$

Since $F' \leq F$ and $b_V e_V(F') = b_V(F'(V)) \leq F$, it follows that the outer diagram and diagram 1 commute, therefore diagram 2 also commutes. This completes the proof of (ii).

(iii) and (iv) follow directly.

(v) To prove (v) assume that $F'(V) \neq F(V)$, then $F'(V) < F(V)$. By (ii) we have $F' \leq b_V e_V(F') = b_V(F'(V))$. Therefore, since $F' \underset{\max}{<} F$, either $F' = b_V(F'(V))$ or $F = b_V(F'(V))$, but the latter possibility can not happen, since $b_V(F(V)) = F$ (by (iii)) and b_V is injective (by (i)). Therefore $b_V(F'(V)) = F'$ or equivalently $b_V e_V(F') = F'$. Now suppose that there exist an $E(V)$-submodule of $F(V)$ with

$$(*) \quad F'(V) < M < F(V) .$$

Since b_V is injective, by applying b_V to (*), we get $b_V e_V(F') < b_V(M) < F$, by (iii). But by (ii) we have $F' \leq b_V e_V(F')$ and so $F' < b_V(M) < F$ which is contradiction since $F' \underset{\max}{<} F$. Therefore $F'(V)$ is maximal $E(V)$-submodule of $F(V)$. This completes the proof of 2.1.1. $\square$

Now let $V \in \text{mod}\,\Lambda$ and let $F = (V\,,\ )$. We have $e_V(F) = F(V) = E(V)$.

Lemma 2.1.2 If $F' < (V,\ )$, then $F'(V)$ is a proper left ideal of $E(V)$.

Proof Suppose $F' < (V,\ )$ with $F'(V) = E(V)$. For any $X \in \text{mod}\,\Lambda$ and $f \in (V,X)$, we have a commutative diagram

$$\begin{array}{ccc} E(V) & \xrightarrow{(V,f)} & (V,X) \\ {\scriptstyle i_V}\uparrow & & \uparrow{\scriptstyle i_X} \\ F'(V) & \xrightarrow{F'(f)} & F(X) \end{array} \qquad \text{(here } (V,f) = (V,\ )(f))$$

Therefore (V,f) maps $F'(V)$ into $F'(X)$, hence $(V,f)(1_{E(V)})$ $(=f) \in F'(X)$. Since that is true for arbitrary $X \in \text{mod}\,\Lambda$ and $f \in (V,X)$, we then have $F'(X) = (V,X)$, and so $F' = (V,\ )$, which contradicts our hypothesis. Therefore $F'(V)$ is a proper left ideal of $E(V)$. □

Now we come to a theorem due to M. Auslander which characterizes the simple objects in $\text{Cov}\,\Lambda$.

Theorem 2.1.3 (Auslander [MA], 1.6, p.275)

(i) If $V \in \text{mod}\,\Lambda$, then e_V and b_V induce a bijection:

$$\{F' \underset{\max}{<} (V,\ )\} \overset{e_V}{\underset{b_V}{\rightleftarrows}} \{\text{maximal left ideals of } E(V)\} .$$

(ii) Given any simple object $S \in \text{Cov}\,\Lambda$, there exist $V \in \text{mod}\,\Lambda$ such that $S \cong (V,\ )/F'$ for some $F' \underset{\max}{<} (V,\ )$.

Proof

(i) From lemma 2.1.2 and proposition 2.1.1(v), we have $F' \underset{\max}{<} (V,\ )$

implies $F'(V)$ is maximal left ideal in $(V,V) = E(V)$. Conversely suppose that M is a maximal left ideal of $E(V)$, then $b_V(M)$ is a proper subfunctor, for if $b_V(M) = (V, \ )$, then by 2.1.1(i), $e_V b_V(M) = M = (V,V) = E(V)$, contradiction. To show that $b_V(M) \underset{\max}{<} (V, \ )$, suppose that there exists a subfunctor F' of $(V, \ )$ such that

$$(**) \qquad b_V(M) < F' < (V, \ ) \ .$$

By 2.1.1(i) and 2.1.2, applying e_V to (**) we get $M \leq F'(V) < E(V)$. But since M is a maximal ideal of $E(V)$, it follows that $F'(V) = M$, and so by 2.1.1(ii), we have $F' \leq b_V(F'(V)) = b_V(M)$. Therefore $b_V(M) = F'$ and so $b_V(M)$ is maximal subfunctor of $(V, \ )$, which completes the proof of (i).

To prove (ii) we need the following lemma.

Yoneda's lemma: (see [SM], p.61): If $F \in \operatorname{Cov} \Lambda$ and $V \in \operatorname{mod} \Lambda$, then the map $\eta : ((V, \ ) , F)_{\operatorname{Cov} \Lambda} \to F(V)$, given by $\eta(t) := t_V(1_V)$ is a bijection. Here $t_V : (V,V) \to F(V)$ is the map defined by the natural transformation $t : (V, \ ) \to F$.

□

Proof of 2.1.3(ii) Since $S \neq 0$, there exists $V \in \operatorname{mod} \Lambda$ such that $S(V) \neq 0$, and so by Yoneda's lemma, there is a non-zero morphism $\alpha : (V, \ ) \to S$. But since S is simple, it follows that α is an epimorphism and so $\operatorname{Ker} \alpha$ is maximal subfunctor, therefore we have $S \cong (V, \ )/\operatorname{Ker} \alpha$. This completes the proof of (ii).

□

Now suppose that $L \in \text{mod}\,\Lambda$ is indecomposable. Take $V = L$ in Theorem 2.1.3. Then $E(L)$ is a local ring ([CRII], p.111) and so has precisely one maximal left ideal, namely $\underline{r}(E(L))$ (Jacobson radical). By 2.1.3(i), $(L, \ )$ has a unique maximal subfunctor $\underline{r}(L, \ )$, where $\underline{r}(L, \ ) := b_L(\underline{r}(E(L))$ given by,

$$\begin{aligned}\underline{r}(L, \ )(X) &:= \underline{r}(L,X) \\ &= \{f \in (L,X) \ , \ \ gf \in \underline{r}(E(L)), \ \forall\, g \in (X,L)\} \ .\end{aligned}$$

Therefore $SL := (L, \ )/\underline{r}(L, \ )$ is simple object of $\text{Cov}\,\Lambda$. Conversely, every simple object S of $\text{Cov}\,\Lambda$ has the form SL for some indecomposable Λ-module L, since it is clear that, in 2.1.3(ii), we can assume that $V = L$ is indecomposable, hence $\text{Ker}\,\alpha$ (in the proof of 2.1.3(ii)) will be the unique maximal subfunctor $\underline{r}(L, \ )$. Summarizing the above we have:

<u>Corollary 2.1.4</u> (Auslander, [MA]) Every simple object in $\text{Cov}\,\Lambda$ has the form SL for some indecomposable Λ-module L. □

<u>Proposition 2.1.5</u> Let L be an indecomposable Λ-module.

(i) If X is an indecomposable Λ-module, and $X \not\cong L$, then $SL(X) = 0$.

(ii) If $X = \coprod_{i \in I} X_i$, where X_i is indecomposable Λ-module, for each $i \in I$, then $SL(X) \cong D(L)^n$, where $n = \#\{i \in I \mid X_i \cong L\} :=$ multiplicity of L in X, and $D(L) := E(L)/\underline{r}(E(L))$.

Proof It is clear that (ii) follows directly from (i). To prove (i) it is enough to show that $\underline{r}(L,X) = (L,X)$. By definition $\underline{r}(L,X) = \{f \in (L,X) \mid gf \in \underline{r}(E(L)) \ \forall \ g \in (X,L)\}$. Since L is indecomposable, $E(L)$ is local algebra, and so $\underline{r}(E(L))$ consists of the non-isomorphisms in $E(L)$ ([CR], p.111).

Hence we have

$$r(L,X) = \{f \in (L,X) | gf \text{ is non-isomorphism for all } g \in (X,L)\} .$$

Given $0 \neq f \in (L,X)$, f can not be isomorphism, since $L \not\cong X$. Let g be any element of (X,L) . The result will follow if we show that gf is non-isomorphism. If gf is an isomorphism, then by triangle lemma (see appendix, lemma 2) we have

$$X \cong \operatorname{Im} f \oplus \operatorname{Ker} g ,$$

where $\operatorname{Im} f :=$ Image of f and $\operatorname{Ker} g =$ kernel of g . But since $\operatorname{Im} f \neq 0$, and X is indecomposable, we then have $\operatorname{Im} f \cong X$, which contradicts the assumption that $X \not\cong L$. Therefore $gf \in E(L)$ is non-isomorphism for all $g \in (X,L)$ and so $r(L,X) = (L,X)$. This completes the proof of 2.1.5(ii).

□

Proposition 2.1.6 ([MA], 1.4(b)) Suppose that S is a simple object in $\operatorname{Cov} \Lambda$, and let $X \in \operatorname{mod} \Lambda$. If $S(X) \neq 0$, then $S(X)$ is simple $E(X)$-module.

Proof By 2.1.3(ii), $S \cong (X, \ )/F'$, for some maximal subfunctor F' of $(X, \ )$. Therefore we have a short exact sequence

2.1.7 $\qquad 0 \to F' \to (X, \ ) \to S \to 0$

in Cov Λ . Applying the evaluation functor e_X to the sequence 2.1.7, and since e_X is exact functor (see [G1], §1), we get a short exact sequence

$$0 \to F'(X) \to (X,X) \to S(X) \to 0$$

in mod $E(X)$. By 2.1.3(i) $F'(X)$ is a maximal left ideal of $(X,X) = E(X)$, and so $S(X) \cong (X,X)/F'(X)$ is simple $E(X)$-module. $\square$

Now suppose that

2.1.8 $\qquad V = V_1 \oplus \dots \oplus V_t$

is a decomposition of $V \in \text{mod}\,\Lambda$ into a direct sum of indecomposable Λ-modules. For each $i \in \{1,\dots,t\}$, let $\mu_i : V_i \to V$ be the inclusion map, and $\pi_i : V \to V_i$ be the projection of V onto V_i . Putting $e_i = \mu_i\pi_i$, it is well-known that $1_{E(V)} = \sum_{i=1}^{t} e_i$ is the orthogonal idempotent decomposition of $1_{E(V)}$ corresponding to the decomposition 2.1.8 of V .

The following theorem relates the simple functors SV_i $(i = 1,2,\dots,t)$ with the simple quotient functors of $(V,\ )$ described in theorem 2.1.3.

Theorem 2.1.9 (Auslander): Suppose $V \in \text{mod}\,\Lambda$ has a decomposition 2.1.8, and let M be any maximal left ideal of $E(V)$. Then

(i) There is at least one $i \in \{1,2,\dots,t\}$ such that $e_i \notin M$.

(ii) If $e_i \notin M$, then $(V,\ )/b_V(M) \cong SV_i$.

Proof

(i) If $e_1,e_2,\dots,e_t$ are all in M , then $1_{E(V)} = \sum_{i=1}^{t} e_i \in M$, which would imply $M = E(V)$.

(ii) Assume $e_i \notin M$. By 2.1.3(i), $b_V(M)$ is a maximal subfunctor of $(V,\ )$. We have $((V,\ )/b_V(M))(V_i) = (V,V_i)/b_V(M)(V_i)$, and $b_V(M)(V_i) = \{f \in (V,V_i) \mid gf \in M\ ,\ \forall\ g \in (V_i,V)\}$. But since $\mu_i\pi_i = e_i \notin M$, it follows that $\pi_i \notin b_V(M)(V_i)$, hence $(V,V_i)/b_V(M)(V_i) = ((V,\ )/b_V(M))(V_i) \neq 0$. Therefore, using 2.1.3(i) and 2.1.5(i), we conclude that $(V,\ )/b_V(M) \cong SV_i$.

□

§2.2 The functors defined by Green

Let V , $X \in \mathrm{mod}\,\Lambda$. In [Gl], Green defined three functors,

$$\ell_{V,X}\ ,\ r_{V,X}\ ,\ q_{V,X} : \mathrm{mod}\,E(V) \to \mathrm{mod}\,E(X)$$

which connect the representation of the k-algebras $E(V)$ and $E(X)$, those functors are explained through the following diagram

2.2.1

$$\begin{tikzcd} \mathrm{Cov}\,\Lambda \arrow[dr, "e_X"'] & & \mathrm{mod}\,E(V) \arrow[ll, bend right=40, "\ell_V"'] \arrow[ll, "q_V"] \arrow[ll, bend left=40, "r_V"] \arrow[ll, phantom, shift right=3, "\Gamma_V"] \arrow[dl] \\ & \mathrm{mod}\,E(X) & \end{tikzcd}$$

$j_{V,X} := e_X \cdot j_V$, $j = \ell, r$ or q

where if M mod $E(V)$, then

$$\ell_V(M) := (V,\ ) \underset{E(V)}{\otimes} M$$

$$r_V(M) := ((\ ,V)\ ,\ M)_{E(V)}$$

and the natural transformation $\Gamma_V : \ell_V \to r_V$ is defined as follows: Given $M \in \text{mod } E(V)$ and $X \in \text{mod } \Lambda$ the map:

2.2.2 $$\Gamma_V(M)(X) : (V,X) \underset{E(V)}{\otimes} M \to ((X,V),M)_{E(V)}$$

is given by: $g \otimes m \to \{f \to (fg)m\}$ for all $g \in (V,X)$, $m \in M$, and $f \in (X,V)$.

The functor $q_V : \text{mod } E(V) \to \text{Cov } \Lambda$ is defined as a quotient functor of ℓ_V as follows:

$$q_V := \ell_V/\text{Ker } \Gamma_V \; .$$

Let j_V be one of the functors

$$\ell_V, r_V, q_V : \text{mod } E(V) \to \text{Cov } \Lambda$$

and for $X \in \text{mod } \Lambda$, let $j_{V,X} := e_X \cdot j_V : \text{mod } E(V) \to \text{mod } E(X)$ (thus $j_{V,X}(M) := j_V(M)(X)$, $\forall\, M \in \text{mod } E(V))$.

<u>Theorem 2.2.3</u> (Green [G1], Thm. 1.9, Prop. 5.2) Suppose $M \in \text{mod } E(V)$ is simple:

(i) $q_V(M)$ is simple in $\text{Cov } \Lambda$.

(ii) If $q_{V,X}(M)$ is not zero, $\text{Ker } \Gamma_{V,X}(M)$ is the unique maximal $E(X)$-submodule of $\ell_{V,X}(M)$ and $\text{Im } \Gamma_{V,X}(M)$ is the unique minimal $E(X)$-submodule of $r_{V,X}(M)$ where $\Gamma_{V,X}(M)$ is given by 2.2.2. It follows that $q_{V,X}(M)$ is a simple $E(X)$-module. □

From now on we will assume that M is a one-dimensional $E(V)$-module (hence simple). Let $\psi : E(V) \to k$ be the one-dimensional character of

$E(V)$ afforded by M, and write $M = k_\psi$, which means k regarded as $E(V)$-module with $E(V)$ acting according to ψ (i.e. $h.c = \psi(h).c$ for every $h \in E(V)$, $c \in k$).

Definition 2.2.4 (Green, [G1], p.266)

(i) For any right $E(V)$-module P, define $A_\psi(P) := \sum_{h \in E(V)} P(h-\psi(h)1_V)$ and $I_\psi(P) := \{p \in P \mid ph = \psi(h)p , \forall\, h \in E(V)\}$. $P/A_\psi(P)$ and $I_\psi(P)$ are the largest quotient module and submodule of P, respectively, on which $E(V)$ acts according to ψ.

(ii) Dual functors: Define a functor $D : \mathrm{Fun}\,\Lambda \to \mathrm{Cov}\,\Lambda$ as follows: If $F \in \mathrm{Fun}\,\Lambda$ we let DF be the object of $\mathrm{Cov}\,\Lambda$ given by $DF(X) := D(F(X))$ for all $X \in \mathrm{mod}\,\Lambda$, where D means the "usual" dual of vector spaces. Note that if $F(X)$ is an A-module for a finite dimensional k-algebra A, then $D(F(X))$ is naturally a right A-module.

It turns out (see [G1] p.264) that when $M = k_\psi$, the functor $\ell_V(k_\psi)$ is isomorphic to a quotient functor of $(V,\ )$ and $r_V(k_\psi)$ is isomorphic to a subfunctor of $D(\ ,V)$ as the following proposition shows.

Proposition 2.2.5 ([G1], Prop. 5.10): Suppose $V, X \in \mathrm{mod}\,\Lambda$, and let $\psi : E(V) \to k$ be a multiplicative character of $E(V)$. Then

(i) $\ell_V(k_\psi) \cong (V,\ )/A_\psi(V,\ )$,

$r_V(k_\psi) \cong I_\psi(D(\ ,V))$,

$q_V(k_\psi) \cong (V,\ )/U_\psi(V,\ )$,

where $A_\psi(V,\ )$ and $U_\psi(V,\ )$ are subfunctors of $(V,\ )$ given by: $A_\psi(V,\ )(X) := A_\psi(V,X)$, $U_\psi(V,\ )(X) := U_\psi(V,X) := \{g \in (V,X) \mid \psi(fg) = 0$, for all $f \in (X,V)\}$, and $I_\psi(D(\ ,V))(X) = I_\psi(D(X,V))$, for all X mod Λ, where (V,X) $[(X,V)]$ are regarded as right [left] $E(V)$-modules.

(ii) The map $\Gamma_{V,X}(k_\psi) : \ell_{V,X}(k_\psi) \to r_{V,X}(k_\psi)$, defined in 2.2.2, induces a map

2.2.7 $\quad \Omega : (V,X) \to D(X,V)$, given by:

$\Omega(f) := \{g \to \psi(gf)\}$, for all $f \in (V,X)$, $g \in (X,V)$, and $\operatorname{Im} \Omega \cong q_{V,X}(k_\psi)$. □

Notice that (V,X) and $D(X,V)$ are (both) right $E(V)$-modules and left $E(X)$-modules, and that the map Ω of 2.2.7 is an $E(V)$-map and an $E(X)$-map (since Ω is natural in both V and X) .

Now suppose that $1_{E(V)} = e_1 +\ldots+ e_t$ is an orthogonal idempotent decomposition of $1_{E(V)}$ in $E(V)$, then there is a unique $j \in \{1,\ldots,t\}$ such that $\psi(e_j) = \psi(1_{E(V)}) = 1_k$; put such $e_j = e_\psi$, and let $e_\psi(V) = V_\psi$ (i.e., V_ψ is the indecomposable direct summand of V which corresponds to the one-dimensional character ψ). Consider the subfunctor $U_\psi(V,\ )$ of $(V,\ )$ defined in 2.2.5

$$U_\psi(V,X) := \{f \in (V,X) \mid \psi(gf) = 0 \ \ \forall f \in (X,V)\}$$

$$= \{f \in (V,X) \mid gf \in \operatorname{Ker} \psi \ \ \forall f \in (X,V)\}$$

putting $\operatorname{Ker} \psi = M_1$, M_1 is a maximal ideal of $E(V)$ and we have

2.2.8 $$U_\psi(V,X) = \{f \in (V\ ,X) \mid (V,g)(f) \in M_1 \ , \ \forall g \in (X,V)\}$$

$$= b_V(M_1)(X) \qquad \text{(see §2.1).}$$

2.2.8 together with 2.2.5 and 2.1.9 and the fact that $e_\psi \notin M_1$ (since $\psi(e_\psi) = 1_k \neq 0$) imply the following:

Lemma 2.2.9 Let V and ψ be as above. Then $q_V(k_\psi) \cong SV_\psi$, where V_ψ is the indecomposable direct summand of V which corresponds to the one-dimensional character ψ. □

The Group algebra case: Now let $\Lambda = kG$, the Group algebra of a finite group G. If X is a subset of G, we write $[X] = \sum_{x \in X} x$. Let H be a subgroup of G, and let $V = kG[H]$. We will see that this will simplify the functors $\ell_V(k_\psi)$, $r_V(k_\psi)$, $q_V(k_\psi)$ and the map $\Omega : (V,X) \to D(X,Y)$ defined in 2.2.7 will have a better shape (see [Gl], p.267). The kG-module $V = kG[H]$ has a k-basis $\{t[H],\ t \in T\}$ where T is a transversal for the set $\{gH,\ g \in G\}$. An element g of G acts on $t[H]$ by the rule:

$$g.t[H] := gt[H] \quad .$$

We take 1 $(\in G)$ to be representative of the left coset H. Every element of $kG[H]$ has the form $\sum_{t \in T} c_t t[H]$, $c_t \in k$, and it is clear that $V = kG[H]$ is isomorphic to the induced kG-module k_H^G from the trivial kH-module k_H.

It is well-known (see [CRII], §11D) that the endomorphism algebra $E(V)$ has a k-basis $\{A_D \mid D \in H\backslash G/H\}$ indexed by the set $H\backslash G/H$ of all double Cosets HgH in G, where $A_D \in E(V)$ is given by:

$$A_D([H]) := a_D[H], \quad \text{where} \quad a_D = \sum_{\substack{t \in T \\ tH \subseteq D}} t \quad .$$

From "Frobenius reciprocity" theorem for modules (see [CRI] p.232) it follows that:

$$(V,\ )(X) := (V,X)_{kG} \cong (k_H,X)_{kH} \quad ,$$

and

$$(\ ,V)(X) := (X,V)_{kG} \cong (X,k_H)_{kH} \quad .$$

Therefore, there is a commutative diagram: (see [G1], p.269)

$$\begin{array}{ccc} (V,X) & \xrightarrow{\ \Omega\ } & D(X,V) \\ \gamma_X \downarrow & & \downarrow \pi_X \\ I_H(X) & \xrightarrow{\ \Omega_1\ } & X/A_HX \end{array} \quad ,$$

where $I_H(X) := \{x \in X \mid hx = x \ , \ \forall\ h \in H\}$,

and $A_HX := \sum_{h\in H} (h-1)X$. The maps γ_X and π_X are k-isomorphisms defined as follows: $\gamma_X : (V,X) \to I_H(X)$ is given by

$$\gamma_X(f) := f([H]) \quad \text{for all} \quad f \in (V,X) \ .$$

To define $\pi_X : D(X,V) \to X/A_HX$, let $\delta : V \to k$ be the kH-map defined by $\delta\ (\sum_t c_t t[H]) = c_1$, then the k-map π_X is defined as follows: for $\theta \in D(X,V)$, $\pi_X(\theta) = x + A_HX$, $x \in X$, if and only if $\theta(f) = \delta f(x)$ for all $f \in (X,V)$. Finally the map $\Omega_1 : I_H(X) \to X/A_HX$ is given by:

2.2.11 $\Omega_1(x) := ax + A_HX$

where a is any element of kG which satisfies the following:

2.2.12 If $a = \sum_{g\in G} c_g\, g^{-1}$, then $\sum_{g\in D} c_g = \psi(A_D)$, for all $D \in H\backslash G/H$.

Since (V,X) and $D(X,V)$ are both right $E(V)$-modules and since γ_X , π_X are k-isomorphisms it follows that $I_H(X)$ and X/A_HX both have a structure of right $E(V)$-module. The $E(V)$-actions on $I_H(X)$ and X/A_HX (denoted by $\circ$) were described in ([G1], p.268) as follows: Let S be a transversal of the set $\{Hg \mid g \in G\}$, thus $G = \dot{\bigcup}_{s\in S} Hs$. Then for every $D \in H\backslash G/H$

2.2.13
$$x \circ A_D = a_D x \quad \text{for all} \quad x \in I_H(X)$$
and
$$(x + A_HX) \circ A_D = a'_D x + A_HX \qquad x \in X$$

where $a'_D = \sum_{\substack{s\in S\\ Hs\subseteq D}} s$ and $a_D = \sum_{\substack{t\in T\\ tH\subseteq D}} t$.

Remark 2.2.14 Since the kG-module X is a left $E(X)$-module, in a natural way (i.e. $fx := f(x)$, for all $f \in E(X)$, $x \in X$) , it follows that both X/A_HX and $I_H(X)$ are left $E(X)$-modules, and it is not difficult to see that the maps γ_X , and π_X of 2.2.10 are both $(E(X),E(V))$-isomorphisms.

The following proposition summarizes the above argument.

Proposition 2.2.15 ([G1], Prop. 5.21) Let $V = kG[H]$, $H \leq G$, let $\psi : E(V) \to k$ be a multiplicative character of $E(V)$, let X be any kG-module, then:

(i) $I_H(X)$ and X/A_HX are $(E(X),E(V))$-bimodules with $E(V)$ and $E(X)$ acting according to 2.2.13, and 2.2.14, respectively. The $E(V)$-action on these sets commutes with the $E(X)$-action.

(ii) $\ell_{V,X}(k_\psi) \cong I_H(X)/A_\psi I_H(X)$,

$r_{V,X}(k_\psi) \cong I_\psi(X/A_H X)$.

(iii) The map $\Gamma_{V,X}(k_\psi) : \ell_{V,X}(k_\psi) \to r_{V,X}(k_\psi)$ is induced by the map $\Omega_1 : I_H(X) \to X/A_H X$, given by 2.2.11.

(iv) $q_{V,X}(k_\psi) \cong I_H(X)/\mathrm{Ker}\ \Omega_1 \cong \mathrm{Im}\ \Omega_1$. □

Notice that, by theorem 2.2.3, $q_{V,X}(k_\psi)$ is a simple left $E(X)$-module, if it is not zero.

In §2.3 we will concentrate on the left $E(X)$-module $r_{V,X}(k_\psi)$. In particular we will assume that $X = {}_{kG}kG$, the regular left kG-module. Since $E(kG) \cong (kG)^{op}$, the opposite ring of kG , the set $I_H(kG)$ and $kG/A_H(kG)$ will have the structure of right $(E(V),kG)$-modules. In each case the right $E(V)$-action commutes with the right kG-action, by 2.2.15(i).

§2.3 <u>The kG-module</u> $r_{kG[H],kG}(k_\psi)$:

<u>Lemma 2.3.1</u> If $V = kG[H]$, $H \le G$, and $X = {}_{kG}kG$, then $X/A_H X$ and $I_H(X)$ defined in 2.2.10 are isomorphic as right $(E(V),kG)$-modules.

<u>Proof</u> It is clear that $I_H({}_{kG}kG) = [H]kG$. Define the map $\zeta : kG \to [H]kG$ by $\zeta(a) := [H]a$ $(a \in kG)$. It is clear that ζ is a right kG-epimorphism with $\mathrm{Ker}\ \zeta = A_H kG$, therefore $kG/A_H kG \cong [H]kG$ as right kG-modules. It remains to show that the map $\zeta_1 : kG/A_H kG \to [H]kG$, induced from ζ , is an $E(V)$-map. Let $D \in H\backslash G/H$, and let $a \in kG$. Then we have

$$\zeta_1((a + A_H kG) \circ A_D) = \zeta_1(a_D' a + A_H kG) = [H]a_D' a$$

$$= [D]a = a_D[H]a = [H]a \circ A_D = \zeta(a + A_H kG) \circ A_D .$$

Hence ζ_1 is an $E(V)$-map. □

From Proposition 2.2.15(ii) and Lemma 2.3.1, we see that:

2.3.2 $$r_{kG[H],kG}(k_\psi) \cong I_\psi(I_H(kG)) = I_\psi([H]kG)$$
$$= \{x \in [H]kG \mid x \circ A_D = \psi(A_D)x \text{ , for all } D \in H\backslash G/H\} .$$

It is a consequence of theorem 2.2.3 that if $q_{kG[H],kG}(k_\psi)$ is not zero, then it is a simple right kG-module, and is isomorphic to the socle of the right kG-module $r_{kG[H],kG}(k_\psi)$.

The next lemma follows from 2.1.5(i) and 2.1.9.

<u>Lemma 2.3.3</u> If $V = kG[H]$, and $\psi : E(V) \to k$ is a multiplicative character of $E(V)$ then $q_{kG[H],kG}(k_\psi) \neq 0$ if and only if V_ψ is a projective kG-module. □

<u>Notations</u>: If $V = kG[H]$, we denote by V' the right kG-module $[H]kG$ and if $E = E(V)$, we denote by E' the endomorphism algebra $\mathrm{End}_{kG}(V')$.

The k-algebra E' has a k-basis $\{A_D' , D \in H\backslash G/H\}$, where $A_D' \in E'$ is given by

$$A_D'([H]) := [D] = [H]a_D' \qquad (a_D' \in kG) .$$

This action will turn $V' = [H]kG$ naturally into left E'-module.

On the other hand V' $(= I_H({}_{kG}kG))$ is also a right E-module (see 2.2.15(i)) by the E-action

$$x \circ A_D := a_D.x \quad \text{for all} \quad x \in V' .$$

Therefore V' is an (E',E)-bimodule with

$$[H] \circ A_D = a_D[H] = [D] = [H]a'_D = A'_D([H]) .$$

Since both actions commute with the G-action on V', we have:

2.3.4 $\qquad x \circ A_D = A'_D(x) \quad \text{for all} \quad x \in V' , \quad D \in H\backslash G/H .$

Let $\theta : E \to E'$ be the k-map given by $\theta(A_D) := A'_D$, for all $D \in H\backslash G/H$. θ is a k-isomorphism which is a k-algebra anti-isomorphism, for if D_1 , $D_2 \in H\ G/H$ and $x \in V'$ then

$$\theta(A_{D_1}A_{D_2})(x) = x \circ A_{D_1}A_{D_2} = (x \circ A_{D_1}) \circ A_{D_2}$$
$$= A'_{D_1}(x) \circ A_{D_2} = A'_{D_2}A'_{D_1}(x) .$$

Hence $\qquad \theta(A_{D_1}A_{D_2}) = \theta(A_{D_2})\theta(A_{D_1})$.

Using the isomorphism θ , equation 2.3.4 will be extended to give

2.3.5 $\qquad x \circ A = \theta(A)(x) \quad \text{for all} \quad x \in V' \quad \text{and all} \quad A \in E .$

<u>Notations 2.3.6</u>: Let V , V' , E , E' and θ be as above.

If $\psi : E \to k$ is a multiplicative character, we denote by $\psi' : E' \to k$ the (unique) multiplicative character of E' which makes the diagram

$$\begin{tikzcd} E \arrow[rr, "\theta"] \arrow[dr, "\psi"'] & & E' \arrow[dl, "\psi'"] \\ & k & \end{tikzcd}$$

commutes. Thus we have

2.3.7 $\psi(A'_D) := \psi(\theta^{-1}(A'_D)) = \psi(A_D)$

for all $D \in H\backslash G/H$.

Now back to the kG-module $r_{V,kG}(k_\psi)$, where $V = kG[H]$. It follows from 2.3.2, 2.3.5, and 2.3.7 that

$$\begin{aligned} r_{V,kG}(k_\psi) &\cong I_\psi(V') \\ &= \{x \in V' \mid x \circ A_D = \psi(A_D)x \text{ , for all } D \in H\backslash G/H\} \\ &= \{x \in V' \mid A'_D(x) = \psi'(A'_D)x \text{ , } \forall\, D \in H\backslash G/H\} \text{ .} \end{aligned}$$

Summarizing the above we have:

Lemma 2.3.8 Suppose $V = kG[H]$, and $V' = [H]kG$ let $\psi : E \to k$ be a multiplicative character of E , and let $\psi' : E' \to k$ be the corresponding multiplicative character of E' according to 2.3.6. Then V' has the structure of (E',E)-bimodule and

$$I_\psi(V') = \{x \in V' \mid A'_D(x) = \psi'(A'_D).x \quad \forall\, D \in H\backslash G/H\} \text{ .} \quad \square$$

Convention 2.3.9 We make the convention that if X is an (E',E)-bimodule and $\psi : E \to k$ is a multiplicative character then we denote by $I_{\psi'}(X)$ $[I_\psi(X)]$ the maximal E' - [E-] submodule of X on which E' [E] acts on the left [right], respectively, according to ψ .

It follows from 2.3.8 and 2.3.9 that

2.3.10 $r_{V,kG}(k_\psi) \cong I_\psi(V') = I_{\psi'}(V')$.

Remark 2.3.11: If we had worked with the category mod'kG of right kG-modules instead of mod kG , we would replace V by V' , ψ by ψ' , and ${}_{kG}kG$ by kG_{kG} . In that case 2.3.10 will give

2.3.12 $$r_{V',kG_{kG}}(k_{\psi'}) \cong I_\psi(V) .$$

Notice that $r_{V',kG_{kG}}(k_{\psi'})$ is a left $E(kG_{kG})$-module, hence (since $E(kG_{kG}) \cong kG$) it can be regarded as a left kG-module. Therefore, and for the sake of notations, we will be dealing with the left kG-module $r_{V',kG_{kG}}(k_{\psi'})$ $(\cong I_\psi(V))$. We will translate all the previous results (proved using mod kG) accordingly. In particular, the right kG-map

$$\Omega_1 : I_H({}_{kG}kG) \ (= [H]kG) \to kG/A_H(kG) \quad ,$$

defined by 2.2.11 will give a left kG-map

$$\Omega_1' : kG[H] \to kG/A_H(kG)$$

defined by

2.3.13 $$\Omega_1'(x) := xa + A_H(kG) \quad \text{for all} \quad x \in kG[H] ,$$

where $a \in kG$ is given by 2.2.12. Similarly we will have a left kG-isomorphism $\zeta_1' : kG/A_H(kG) \to kG[H]$ (corresponding to the map ζ_1 defined in the proof of 2.3.1) given by

2.3.14 $$\zeta_1'(x + A_H(kG)) := x[H] \quad \text{for all} \quad x \in kG .$$

Lemma 2.3.15 Let V be any kG-module, and let $1_{E(V)} = e_1 + \ldots + e_t$ be an orthogonal primitive idempotent decomposition of $1_{E(V)}$ in $E(V)$.

If $\psi : E(V) \to k$ is a multiplicative character of $E(V)$, we write $V_\psi = e_\psi(V)$, where e_ψ is the unique e_i with $\psi(e_i) = 1_k$. Then $I_\psi(V) \leq V_\psi$.

<u>Proof</u> If $x \in I_\psi(V)$, then

$$x = 1_k . x = \psi(e_\psi)x = e_\psi(x) \in e_\psi(V) = V_\psi \quad .$$

Therefore $I_\psi(V) \leq V_\psi$. $\square$

CHAPTER 3. The Steinberg Representation

In this chapter we let G be a finite group with a BN-pair whose coxeter system is $(W,\underline{R})$ with $|\underline{R}| = \ell$. Let $V = kG[B]$ where B is the standard Borel subgroup of G . Let $E = \mathrm{End}_{kG}(kG[B])$; it is clear from 1.1.2 that E has a k-basis indexed by the elements of W , in fact $E = \sum_{w \in W}^{\oplus} k.A_w$, where $A_w \in E$ is given by $A_w([B]) := [BwB]$. It is well-known (see [I]) that E is generated as a k-algebra by the set $\{A_{w_i}, w_i \in \underline{R}\}$.

Let $\psi : E \to k$ be the multiplicative character, defined by Iwahori [I], where $\psi(A_w) := (-1)^{\ell(w)}$, and $\ell(w)$ is the length of w .

In this chapter, we will describe the kG-module $I_\psi(V)$, defined in §2, consequently (§3.2), we recover some well-known results about Steinberg representation for finite groups with BN-pair. Also the study of the kG-module $I_\psi(V)$ will provide a new characterization of the Steinberg representation. The essential tool which will be used is a simplicial complex (defined by Tits), associated with an arbitrary finite group with BN-pair.

§3.1 The Tits Complex ([JT])

To every finite group with BN-pair $(G,B,N,\underline{R})$, there is associated a certain simplicial complex Δ called the Tits complex (see for example [LS]), described as follows:

For each $i = 1,2,\ldots,\ell$, let $J_i = \underline{R}\backslash\{w_i\}$ and let $G_i = G_{J_i} = BW_{J_i}B$. It is clear that

3.1.1 $\qquad G_1 , G_2 , \ldots, G_\ell$

are maximal parabolic subgroups of G which contain B ; we shall suppose that the order of the G_i's is fixed as in 3.1.1. If P is any maximal parabolic of G then P contains the Borel subgroup B^g for some $g \in G$, therefore $B \leq {}^gP$ and hence ${}^gP = G_j$ for some $j \in \{1,2,\ldots,\ell\}$.

Definition 3.1.2 We say that P (above) is of type j .

Remark 3.1.3: there is no ambiguity in definition 3.1.2, for if $B^h \leq P$ for some $h \in G$, then ${}^gP$ and ${}^hP$ are conjugate maximal parabolics containing the same Borel subgroup and hence, by Tit s theorem (1.1.4(2)), ${}^gP = {}^hP = G_j$.

Convention 3.1.4: We make the convention that if P_j is a maximal parabolic subgroup then j = type of P_j , i.e. the suffix indicates the type.

The Tits complex of G has the set of all maximal parabolics of G as its vertices, and if $0 \leq r \leq \ell-1$ then the $r+1$ vertices $P_{j_0}, P_{j_1} ,\ldots,P_{j_r}$ form an r-simplex (written as $(P_{j_0},\ldots, P_{j_r})$) if and only if $P_{j_0} \cap P_{j_1} \cap\ldots\cap P_{j_r}$ is a parabolic subgroup of G , and $j_0 < j_1 < \ldots <j_r$. For $0 \leq r \leq \ell-1$, denote by Δ_r the set of r-simplexes, i.e. $\Delta_r = \{(P_{j_0},\ldots,P_{j_r}) \mid \bigcap_i P_{j_i}$ is parabolic, and $j_i < j_{i+1}$ for all $i = 0,\ldots,r\}$. Let $C_r = k\Delta_r$ be a k-vector space with Δ_r as basis, for $0 \leq r \leq \ell-1$. We define $C_r = 0$ if $r \geq \ell$

or if $r \leq -1$. G acts on the vertices of Δ by conjugation. This action on Δ preserves the simplicial structure and hence will turn C_r , $0 \leq r \leq \ell-1$, into a kG-module where,

$$g \cdot (P_{j_0}, \ldots, P_{j_r}) := ({}^gP_{j_0}, \ldots, {}^gP_{j_r}) ,$$

for every $g \in G$ and every $(P_{j_0}, \ldots, P_{j_r}) \in \Delta_r$.

Define the k-map $\partial_r : C_r \to C_{r-1}$ $(1 \leq r \leq \ell-1)$ by

3.1.5 $$\partial_r((P_{j_0},\ldots,P_{j_r})) := \sum_{\rho=0}^{r} (-1)^\rho (P_{j_0},\ldots,P_{j_{\rho-1}}, P_{j_{\rho+1}},\ldots,P_{j_r}) ,$$

extended by linearity. It is clear that ∂_r is a kG-map with $\partial_r \partial_{r+1} = 0$ for all $1 \leq r \leq \ell-1$, hence the pair (Δ,∂) defines a simplicial complex.

Definition 3.1.6 ([R], p.20): The r-th homology group $H_r(\Delta)$ of Δ is the r-th homology group of the complex

$$0 \to C_{\ell-1} \xrightarrow{\partial_{\ell-1}} C_{\ell-2} \xrightarrow{\partial_{\ell-2}} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0 \to \cdots$$

i.e. $$H_r(\Delta) := \operatorname{Ker} \partial_r \,/\, \operatorname{Im} \partial_{r+1} .$$

Now consider the kG-module $C_{\ell-1}$. The $(\ell-1)$-simplexes are called chambers (see [LS]) and the chamber $\mathcal{B} = (G_1,\ldots,G_\ell)$ is called the fundamental chamber.

Note that if $\mathcal{P} = (P_{j_0}, \ldots, P_{j_{\ell-1}})$ is a chamber then $\bigcap_i P_{j_i}$ is a Borel subgroup.

Lemma 3.1.7

(i) If $g \in G$ then, $g\mathcal{B} = \mathcal{B}$ if and only if $g \in B$.

(ii) $C_{\ell-1}$ is a cyclic kG-module generated by $\mathcal{B}$ and is isomorphic to $kG[B]$.

Proof

(i) If $g \in G$, then $g\mathcal{B} = g(G_1,\ldots,G_\ell) = ({}^gG_1,\ldots,{}^gG_\ell)$. Therefore $g\mathcal{B} = \mathcal{B}$ if and only if $g \in \bigcap_i N_G(G_i) = \bigcap_i G_i = B$, by 1.1.3, where $N_G(G_i)$ is the normalizer of G_i in G .

(ii) Suppose that $P = (P_{j_0},\ldots, P_{j_{\ell-1}}) \in \Delta_{\ell-1}$. Then we have $P_{j_0} \cap P_{j_1} \cap\ldots\cap P_{j_{\ell-1}} = {}^gB$ for some $g \in G$. Therefore,

$$(P_{j_0},\ldots, P_{j_{\ell-1}}) = ({}^gG_1,{}^gG_2,\ldots,{}^gG_\ell) = g.(G_1,\ldots,G_\ell)$$
$$= g\mathcal{B} \ .$$

Since $C_{\ell-1}$ is the k-span of the elements of $\Delta_{\ell-1}$, $C_{\ell-1} = \sum^{\oplus}_{P\in\Delta_{\ell-1}} k.P = kG.\mathcal{B}$ and the map $\mathcal{B} \to [B]$ defines a kG-isomorphism $\alpha : C_{\ell-1} \to kG[B]$.

□

Since the kG-module $V = kG[B]$ is a left E-module (where $E = E(V)$) , we can define an E-action on $C_{\ell-1}$ $(= kG.\mathcal{B})$ (this action denoted by $*$) by means of the isomorphism α defined in the proof of 3.1.7(ii). That is, if $A \in E$ and $g \in G$, we let

3.1.8 $\qquad A * g\mathcal{B} := \alpha^{-1}\,(A(g[B]))$.

Since the E-action and the G-action on $V = kG[B]$ commute, we then have:

$$A * g\mathcal{B} = \alpha^{-1}(A(g[B])) = \alpha^{-1}.gA([B]))$$
$$= g\,\alpha^{-1}(A([B])) = g.A * \mathcal{B}.$$

Therefore the action of $A \in E$ on $C_{\ell-1}$ is determined by its action on $\mathcal{B} = (G_1,\ldots,G_\ell)$; it is also clear that $I_\psi(kG[B]) \cong I_\psi(C_{\ell-1})$ as kG-modules.

To describe $I_\psi(C_{\ell-1})$ we need to introduce the following notation.

<u>Notations 3.1.9</u>

(i) If $P = (P_{j_1},\ldots, P_{j_\ell}) \in \Delta_{\ell-1}$ is an $(\ell-1)$-simplex and $\rho \in \{1,\ldots, \ell\}$, we denote by $P^{(\rho)}$ the $(\ell-2)$-simplex obtained from P by removing the ρ-th term, i.e. $P^{(\rho)} = (P_{j_1},\ldots, P_{j_{\rho-1}}, P_{j_{\rho+1}},\ldots, P_{j_\ell})$.

(ii) If P' and P are $(\ell-2)$, $(\ell-1)$-simplexes, respectively, we write $P' < P$ if P' is obtained from P by removing one term.

The G-action on the vertices of Δ induces an action on the homology group $H_{\ell-1}(\Delta)$. We have the following description for the homology module.

<u>Lemma 3.1.10</u> If $x = \sum_{P\in\Delta_{\ell-1}} x(P)P$, with $x(P) \in k$, then $x \in H_{\ell-1}(\Delta)$ if and only if for every $\rho \in \{1,\ldots,\ell\}$ and every $(\ell-2)$-simplex Z , $\sum_{\substack{P \in \Delta_{\ell-1} \\ P^{(\rho)} = Z}} x(P) = 0$.

Proof By 3.1.6, since $\operatorname{Im} \partial_\ell = 0$, we have $H_{\ell-1}(\Delta) = \operatorname{Ker} \partial_{\ell-1}$. put $\partial_{\ell-1} = \partial$. We have

$$\partial x = \sum_{P \in \Delta_{\ell-1}} x(P)\partial P = \sum_{P} x(P)\Big(\sum_{\rho=1}^{\ell} (-1)^{\rho-1} P^{(\rho)}\Big) .$$

Fix $\rho \in \{1,\ldots,\ell-1\}$. If $Z = (Z_{j_1},\ldots,Z_{j_{\rho-1}},Z_{j_{\rho+1}},\ldots,Z_{j_\ell}) \in \Delta_{\ell-2}$, then Z appears in ∂x with coefficient $(-1)^{\rho-1} \sum_{\substack{P \in \Delta_{\ell-1} \\ P^{(\rho)} = Z}} x(P)$.

Therefore $\partial x = 0$ if and only if $\sum_{\substack{P \in \Delta_{\ell-1} \\ P^{(\rho)} = Z}} x(P) = 0$ for all $\rho \in \{1,\ldots,\ell\}$ and all $Z \in \Delta_{\ell-2}$. □

Notations 3.1.11

(i) Let $Z = (Z_{j_1},\ldots,Z_{j_{\rho-1}},Z_{j_{\rho+1}},\ldots,Z_{j_\ell})$ be an $(\ell-2)$-simplex. We denote by $X_\rho(Z)$ the set of all maximal parabolic subgroups X of G such that $(Z_{j_1},\ldots,Z_{j_{\rho-1}},X,Z_{j_{\rho+1}},\ldots,Z_{j_\ell})$ is an $(\ell-1)$-simplex.

(ii) If $w_i \in \underline{R}$ let B_i be a transversal for the set $\{b(B \cap {}^{w_i}B) \mid b \in B\}$.

The following lemma determines the set $X_\rho(Z)$ for $Z = \mathcal{B}^{(\rho)}$ where $\mathcal{B} = (G_1,\ldots,G_\ell)$ is the fundamental chamber.

Lemma 3.1.12 $X_\rho(\mathcal{B}^{(\rho)}) = \{G_\rho, {}^{bw_\rho}G_\rho \; ; \; b \in B_i\}$.

Proof Consider the $(\ell-2)$-simplex $\mathcal{B}^{(\rho)} = (G_1,\ldots,G_{\rho-1},G_{\rho+1},\ldots,G_\ell)$

and let $P = \bigcap_{\substack{i=1 \\ i \neq \rho}}^{\ell} G_i$. It is clear that P is a minimal parabolic subgroup, in fact $P = BW_{\{w_\rho\}}B = B \cup Bw_\rho B$. If X is a maximal parabolic subgroup of G, then $X \in X_\rho(B^{(\rho)})$ if and only if $X \cap P$ is a Borel subgroup. If $B' = X \cap P$, then $B' = {}^gB$ for some $g \in G$. Hence both P and ${}^{g^{-1}}P$ are conjugate parabolic subgroups containing B . Therefore, by 1.1.4(2), $P = {}^{g^{-1}}P$, and so $g \in N_G(P) = P = B \cup B_\rho w_\rho B$. It follows that g is <u>either</u> in B <u>or</u> in $B_\rho w_\rho B$ and $B' \in \{B , {}^{b(w_\rho)}B ; b \in B_\rho\}$. The members of the set $\{B, {}^{b(w_\rho)}B ; b \in B_\rho\}$ are all distinct, for if $b , b' \in B_\rho$, then

${}^{b(w_\rho)}B = {}^{b'(w_\rho)}B \iff (b'(w_\rho))^{-1} b(w_\rho) \in N_G(B) = B \iff b'(w_\rho)B = b(w_\rho)B$
$\iff b = b'$.

It follows that $X \in X_\rho(B^{(\rho)}) \iff X \cap P = {}^gB$, $g \in \{1,b(w_\rho); b \in B_\rho\}$ $\iff {}^{g^{-1}}X \cap {}^{g^{-1}}P = B \iff {}^{g^{-1}}X \cap P = B$ (since $b(w_\rho) \in P = N_G(P)$, for all $b \in B_\rho$) $\iff {}^{g^{-1}}X = G_\rho \iff X = {}^gG_\rho$, $g \in \{1, b(w_\rho) ; b \in B_\rho\}$. Therefore $X \in \{{}^gG_\rho \mid g \in \{1, b(w_\rho) ; b \in B_\rho\}\}$.

The parabolic subgroups $\{{}^gG_\rho \mid g \in \{1, b(w_\rho), b \in B_\rho\}\}$ are all distinct, since they contain distinct Borel subgroups.

It is also clear that these parabolic subgroups are all of type ρ . This completes the proof of the lemma. $\square$

Since $E = E(kG[B])$ is generated as a k-algebra by $\{A_{w_i} , w_i \in \underline{R}\}$, it follows that the action of the elements A_{w_i} $(w_i \in \underline{R})$ on the k-basis of $C_{\ell-1}$ determines the E-action on $C_{\ell-1}$.

<u>Lemma 3.1.13</u> For every $w_i \in \underline{R}$,

(i) $(A_{w_i} + 1_E) * \mathcal{B} = \sum_{\substack{\mathcal{H} \in \Delta_{\ell-1} \\ \mathcal{B}^{(i)} < \mathcal{H}}} \mathcal{H}$.

(ii) If $\mathcal{P} \in \Delta_{\ell-1}$ then

$$(A_{w_i} + 1_E) * \mathcal{P} = \sum_{\substack{\mathcal{P}' \in \Delta_{\ell-1} \\ \mathcal{P}^{(i)} < \mathcal{P}'}} \mathcal{P}' .$$

<u>Proof</u> (ii) follows from (i), since if $\mathcal{P} \in \Delta_{\ell-1}$ then $\mathcal{P} = g\mathcal{B}$ for some $g \in G$, and the E-action commutes with the G-action on $C_{\ell-1}$. To prove (i) consider $A_{w_i} * \mathcal{B}$,

$$\begin{aligned} A_{w_i} * \mathcal{B} &= \alpha^{-1}(A_{w_i}([B])) \\ &= \alpha^{-1}(\sum_{b \in B_i} b(w_i)[B]) \\ &= \sum_{b \in B_i} b(w_i)\alpha^{-1}([B]) = \sum_{b \in B_i} b(w_i)\mathcal{B} \\ &= \sum_{b \in B_i} ({}^{b(w_i)}G_1 , \ldots , {}^{b(w_i)}G_i , \ldots , {}^{b(w_i)}G_\ell) \\ &= \sum_{b \in B_i} (G_1, \ldots , {}^{b(w_i)}G_i , \ldots , G_\ell) \end{aligned}$$

(since if $\rho \neq i$, then $b(w_i) \in G_\rho = N_G(G_\rho))$.

Therefore

$$\begin{aligned} (A_{(w_i)} + 1_E) * \mathcal{B} &= \mathcal{B} + \sum_{b \in B_i} (G_1, \ldots , {}^{b(w_i)}G_i, \ldots , G_\ell) \\ &= \sum_{X \in X_i(\mathcal{B}^{(i)})} (G_1, \ldots , G_{i-1}, X, G_{i+1}, \ldots , G_\ell) \quad \text{by lemma 3.1.12} \\ &= \sum_{\substack{\mathcal{H} \in \Delta_{\ell-1} \\ \mathcal{B}^{(i)} < \mathcal{H}}} \mathcal{H} \end{aligned}$$

□

We now come to the main theorem of this chapter, concerning the kG-module $I_\psi(kG[B])$.

Theorem 3.1.14 Let G be a finite group with BN-pair, and let k be an arbitrary field. Let $V = kG[B]$ and let $E = \mathrm{End}_{kG}(V)$. Let $\psi : E \to k$ be the multiplicative character of E given by $\psi(A_w) = (-1)^{\ell(w)}$, for every $w \in W$. Then, regarding V as a left E-module, we have:

$$I_\psi(kG[B]) \cong H_{\ell-1}(\Delta)$$

where Δ is the Tits complex of G .

Proof We have $I_\psi(kG[B]) \cong I_\psi(C_{\ell-1})$

$$= \{x \in C_{\ell-1} \mid A * x = \psi(A)x \ , \quad \text{for all} \quad A \in E\}$$

$$= \{x \in C_{\ell-1} \mid A_{w_i} * x = -x \ , \quad \text{for all} \quad w_i \in \underline{R}\}$$

$$= \{x \in C_{\ell-1} \mid (A_{w_i} + 1_E)*x = 0 \ , \quad \text{for all} \quad w_i \in \underline{R}\} \quad .$$

Let $x = \sum_{P\in\Delta_{\ell-1}} x(P)P$ be an arbitrary element of $C_{\ell-1}$, with $x(P) \in k$.

Consider $(A_{w_i} + 1_E)*x$ $(w_i \in \underline{R})$. By 3.1.13, we have

$$(A_{w_i} + 1_E)*x = \sum_{P\in\Delta_{\ell-1}} x(P) \quad (A_{w_i} + 1_E)*P$$

$$= \sum_P x(P) \quad \Big(\sum_{\substack{Q\in\Delta_{\ell-1} \\ P^{(i)}<Q}} Q\Big) \quad .$$

Let Q be an arbitrary $(\ell-1)$-simplex with $Q^{(i)} = Z$, for some

$Z \in \Delta_{\ell-2}$. Then Q appears in $(A_{w_i} + 1_E)*x$ with coefficient $\sum_{P^{(i)}=Z} x(P)$; comparing this with lemma 3.1.10, and since that is valid for every $w_i \in \underline{R}$, we have $(A_{w_i} + 1_E)*x = 0$, $\forall\, w_i \in \underline{R} \iff x \in H_{\ell-1}(\Delta)$, therefore

$$I_\psi(kG[B]) \cong I_\psi(C_{\ell-1}) \cong H_{\ell-1}(\Delta) . \qquad \square$$

§3.2 Application to the Steinberg Representation

Let $G = (G,B,N,\underline{R},U)$ be a finite group with split BN-pair of characteristic p , and let k be an algebraically closed field. In [RS] Steinberg defined the kG-module kGe , where $e = \sum_{w \in W} (-1)^{\ell(w)} w[B]$ and showed that this module is absolutely irreducible when the characteristic of k is either zero or does not divide $[G:B]$. Let St_G denote the character afforded by kGe .

<u>Definition</u> kGe is called the Steinberg module, and St_G is called the Steinberg character of G .

In this section, we will apply the results in §2.2 and §3.1 to derive some results about the Steinberg representation. As in §3.1, we let $V = kG[B]$, $E = E(V)$, and $\psi : E \to k$ be the multiplicative character of E given by $\psi(A_{w_i}) = -1$, for all $w_i \in \underline{R}$.

First we need the following lemma which is due to N. Tinberg.

<u>Lemma 3.2.1</u> (Tinberg, [NT1]): $A_{w_i}(e) = -e$ for all $w_i \in \underline{R}$. $\square$

It follows from lemma 3.2.1 that the Steinberg module kGe is a kG-submodule of $I_\psi(V)$, and by 2.3.15, we have

3.2.2 $$kGe \leq I_\psi(V) \leq V_\psi ,$$

where "≤" means kG-submodule. Suppose that the characteristic of k does not divide $|H|$, where $H = B \cap N$. Then V is a direct summand of kG[U] (see [CL], Lemma 3.3). In this case, the indecomposable kG-module V_ψ is a direct summand of kG[U]. Since kG[U] is a projective kG-module for arbitrary field k (see [NT1] p.129), it follows that V_ψ is a projective indecomposable kG-module. Hence V_ψ has a simple socle (see [CRI], p. 401). Therefore, by 3.2.2, we have

$$\mathrm{soc}(kG.e) = \mathrm{soc}(I_\psi(V)) = \mathrm{soc}(V_\psi) ,$$

where "soc" means socle as kG-module.

By remark 2.3.11,

$$I_\psi(V) \cong r_{V',kG_{kG}}(k_{\psi'}) ,$$

where $V' = [B]kG$, and $\psi' : E(V') \to k$ is the multiplicative character of $E(V')$ which corresponds to ψ (see 2.3.6). We can deduce, using similar argument to the above, that $V'_{\psi'}$ is a projective indecomposable right kG-module (see 2.3.11). It follows from 2.3.3 and 2.2.3 that $q_{V',kG_{kG}}(k_{\psi'})$ is non-zero and hence it is the socle of the left kG-module $r_{V',kG_{kG}}(k_{\psi'})$. Therefore

3.2.2 $$\mathrm{soc}(kGe) = \mathrm{soc}(I_\psi(V)) \cong q_{V',kG_{kG}}(k_{\psi'}) .$$

The socle of the Steinberg module kGe was computed by N. Tinberg [NT1] for arbitrary field k, and for a finite group with "unsaturated" split BN-pair. The following proposition provides a similar calculation for $soc(kG.e)$.

Proposition 3.2.3 Let $G = (G,B,N,\underline{R},U)$ be a finite group with a split BN-pair of characteristic p and rank ℓ. Let W be the Weyl group of G and let k be an algebraically closed field such that the characteristic of k does not divide $|B \cap N|$. Then $soc(kG.e) = kG\,[B]\{W\}[B]$, where $\{W\} = \sum_{w \in W} (-1)^{\ell(w)} w$.

Proof From 3.2.2, we have

$$soc(kG.e) = q_{V',kG_{kG}}(k_{\psi'}) .$$

By 2.2.15(iv), we have

$$soc(kG.e) \cong \text{Image of } \Omega_1'$$

where $\Omega_1' : kG[B] \to kG/A_B(kG)$ is given by $\Omega_1'(x) := xa \quad (x \in kG[B])$, and a is any element of kG satisfying 2.2.12 (see remark 2.3.11). On the other hand, by 2.3.1, the map $\zeta_1' : kG/A_B(kG) \to kG[B]$, given by $\zeta_1'(x + A_B(kG)) := x[B] \quad (x \in kG)$, defines a kG-isomorphism. Therefore we have

$$soc(kG.e) = \text{Im } \zeta_1' \Omega_1' .$$

$$\begin{tikzcd} kG[B] \arrow[r, "\Omega_1'"] \arrow[dr, "\zeta_1' \Omega_1'"'] & kG/A_B(kG) \arrow[d, "\zeta_1'"] \\ & kG[B] \end{tikzcd} .$$

However, it is easy to see that $\{W\} = \sum_{w\in W} (-1)^{\ell(w)} w$ satisfies 2.2.12. Hence, putting $a = \{W\}$, we have

$$\zeta_1'\Omega_1'([B]) = [B]\{W\}[B] \quad .$$

Consequently,

$$\mathrm{soc}(kG.e) = \mathrm{Im}\ \zeta_1'\Omega_1' = kG[B]\{W\}[B] \quad .$$

$\square$

<u>The ordinary case</u>:

Now suppose that k is of characteristic 0. Curtis [C1] discovered a formulae which expresses St_G as Z-linear combination of induced characters from the parabolic subgroups of G, in fact he shows that

3.2.4 $$St_G = \sum_{J\subseteq \underline{R}} (-1)^{|J|}\, 1_{G_J}^{G} \quad .$$

This formulae gives a more general setting of Steinberg character for arbitrary finite groups with BN-pairs (see [C1] or [RC], 6.1-6.4). By Maschke's theorem (see [CRII], p.42), every object in mod kG is completely reducible. Therefore, by 2.2.15(iv), we have

$$r_{[B]kG,kG}(k_{\psi'}) \cong q_{[B]kG,kG}(k_{\psi'}) \ ,$$

hence, $r_{[B]kG,kG}(k_{\psi'})$ is simple kG-module. But Theorem 3.1.14 shows that

$$r_{[B]kG,kG}(k_{\psi'}) \cong I_{\psi}(kG[B]) \cong H_{\ell-1}(\Delta) \quad .$$

This gives a proof of the following theorem which has been originally proved by Solomon and Tits [LS], using geometrical argument.

Theorem 3.2.5 (Solomon-Tits): Let Δ be the Tits complex of a finite group G with BN-pair of rank $\ell \geq 2$, and let k be a field of characteristic 0. The action of G on Δ defined a kG-module structure in $H_{\ell-1}(\Delta)$ which affords the Steinberg character.

□

Example

Let $G = SL(2,q) = \{g \in GL(2,q),\ \det g = 1\}$, where q is a power of a prime $p > 0$. G has a split BN-pair $(G,B,N,\underline{R},U)$, where

$$B = \left\{\begin{pmatrix} t & \lambda \\ 0 & t^{-1}\end{pmatrix},\quad t \in \mathbb{F}_q^{\times},\quad \lambda \in \mathbb{F}_q\right\},$$

$$N = \left\{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix},\quad \begin{pmatrix} & -\lambda \\ \lambda^{-1} & \end{pmatrix},\quad \lambda, t \in \mathbb{F}_q^{\times}\right\},$$

$$H = \left\{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix},\quad t \in \mathbb{F}_q^{\times}\right\}, \text{ and}$$

$$U = \left\{\begin{pmatrix} 1 & \lambda \\ & 1\end{pmatrix},\quad \lambda \in \mathbb{F}_q\right\},$$

where $\mathbb{F}_q = GF(q)$, the finite Galois field with q elements, and $\mathbb{F}_q^{\times} = \mathbb{F}_q \backslash \{0\}$. The Weyl group $W = N/H \cong S_2 = \langle w\rangle = \langle\underline{R}\rangle$. We may take $(w) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$; we have $N = H \,\dot{\cup}\, (w)H$ and $G = B \,\dot{\cup}\, BwB$.

Let $M = V(2,q) = \left\{\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix};\ \lambda_1, \lambda_2 \in \mathbb{F}_q\right\}$, be the space of 2-dimensional column vectors over $\mathbb{F}_q$. Let $\sim$ be the equivalence relation defined on the elements of M as follows:

For $x,y \in M$, $x \sim Y \iff \lambda x = y$ for some $\lambda \in \mathbf{F}_q^\times$. The projective line $P_1(\mathbf{F}_q)$ is the set of equivalence classes of the non-zero elements of M under the equivalence relation $\sim$. If $0 \neq x \in M$, let $[x]$ denote the equivalence class which contains x, and if $0 \neq \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \in M$, we write $\left[\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\right] = \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$. Thus, if $0 \neq x \in M$, then

$$[x] = \{\lambda x \mid \lambda \in \mathbf{F}_q^\times\} \quad .$$

Clearly $P_1(\mathbf{F}_q) = \{\infty := \begin{bmatrix}1\\ 0\end{bmatrix}, [\lambda] := \begin{bmatrix}\lambda\\ 1\end{bmatrix}; \lambda \in \mathbb{F}_q\}$. Let k be a field and let $X = k.P_1(\mathbf{F}_q)$, the k-space having the elements of $P_1(\mathbf{F}_q)$ as basis. The group G acts transitively on the set $P_1(\mathbb{F}_q)$. This action can be extended by linearity to turn X into a kG-module with $\dim_k X = q+1$.

If $g = \begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} \in G$, then $g\infty = g\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}g_{11}\\ g_{21}\end{bmatrix}$. Hence

$$\begin{aligned} g\infty = \infty &\iff \begin{bmatrix}g_{11}\\ g_{21}\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \\ &\iff \begin{pmatrix}\lambda g_{11}\\ \lambda g_{21}\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \\ &\iff g_{21} = 0 \iff g \in B \ . \end{aligned}$$

If $\lambda \in \mathbb{F}_q$, then

$$\begin{pmatrix}1 & \lambda\\ & 1\end{pmatrix}(w)\infty = \begin{pmatrix}1 & \lambda\\ & 1\end{pmatrix}[0] = \begin{pmatrix}1 & \lambda\\ & 1\end{pmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}\lambda\\ 1\end{bmatrix} \equiv [\lambda] \quad .$$

Therefore the map $[B] \to \infty$ defines a kG-isomorphism $\alpha : kG[B] \to X$.

Let $E = E(kG[B])$. E has k-basis $\{1, A_w\}$, where $A_w([B]) := \sum_{u \in U} u(w)[B]$ (note that $U_w^- = U$) . The kG-module $kG[B]$ has a (natural) structure of left E-module. Since $kG[B] \cong X$ (= $kG.\infty$) , we can describe an E-action on X (denoted by $*$) in terms of the kG-isomorphism α . That is

$$\begin{aligned} A_w * \infty &= \alpha(A_w([B])) \\ &= \alpha(\sum_{u \in U} uw[B]) \\ &= \sum_{u \in U} uw\alpha([B]) = \sum_{u \in U} uw\infty \\ &= \sum_{u \in U} u[0] = \sum_{\lambda \in \mathbb{F}_q} \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix} [0] \\ &= \sum_{\lambda \in \mathbb{F}_q} [\lambda] \\ &= \sum_{\substack{\sigma \in P_1(\mathbb{F}_q) \\ \sigma \neq \infty}} \sigma \quad . \end{aligned}$$

Since G acts transitively on $P_1(\mathbb{F}_q)$ and since the E-action on X commutes with the G-action, we then have

3.2.6 $$A_w * \sigma = \sum_{\substack{y \in P_1(\mathbb{F}_q) \\ y \neq \sigma}} y \qquad \text{for all } \sigma \in P_1(\mathbb{F}_q) \ .$$

Let $\psi : E \to k^\times$ be the multiplicative character of E given by $\psi(A_w) = -1$. If $x = \sum_{\pi \in P_1(\mathbb{F}_q)} x(\pi)\pi \in X$, $x(\pi) \in k$, then, by 3.2.6, we have

$$A_w * x + x = \sum_{\pi\in P_1(\mathbb{F}_q)} x(\pi)\, A_w * \pi + \sum_{\pi\in P_1(\mathbb{F}_q)} x(\pi)\pi$$

$$= \sum_{\pi\in P_1(\mathbb{F}_q)} x(\pi)\ (A_w * \pi + \pi)$$

$$= \Big(\sum_{\pi\in P_1(\mathbb{F}_q)} x(\pi)\Big)\Big(\sum_{\sigma\in P_1(\mathbb{F}_q)} \sigma\Big) .$$

Therefore

$$I_\psi(kG[B]) \cong I_\psi(X) = \{x \in X \mid (A_w+1)*x = 0\}$$

$$= \{x = \sum_{\pi\in P_1(\mathbb{F}_q)} x(\pi)\pi \in X \mid \sum_{\pi} x(\pi) = 0\}$$

$$\cong \{a = \sum a_g g[B] \in kG[B] \mid \sum_{g\in G} a_g = 0\} .$$

It is clear that $kG\,(1-(w))[B] \subseteq I_\psi(kG[B])$. On the other hand

$$A_\psi(kG[B]) \cong A_\psi(X) = (A_{V'} + 1_E)X$$

$$= kG.(A_w + 1_E) * \infty$$

$$= kG.\Big(\sum_{\pi\in P_1(\mathbb{F}_q)} \pi\Big)$$

$$\cong k_G ,\ \text{ the trivial } kG\text{-module.}$$

Therefore, it follows from 2.2.4(i) that $I_\psi(kG[B]) = kG(1-(w))[B]$.

PART II :

On The Theory Of Modular Representations Of Finite Groups

With Split BN-pairs.

CHAPTER 4. Modular representations of finite Groups with Split BN-pair.

§4.0 Preliminaries and basic definitions

In part II we let $G = (G,B,N,\underline{R},U)$ be a finite group with a split BN-pair of characteristic p, for some prime $p > 0$ (§1.2). Let W be the Weyl group of G generated by the set $\underline{R} = \{w_1,\dots,w_\ell\}$ of simple reflections. Let (K,R,F) be a p-modular coefficient system ([CRII], p.402), that is, R is a p-adic ring with quotient field K of characteristic 0, maximal ideal (π) $(= \pi R)$, and residue class field $F = \bar{R} = R/\pi R$ of characteristic p. We shall assume that K (and hence F) is splitting field for G and all its subgroups, and that R is complete in the (π)-adic topology ([CRII], p.83). An <u>R-lattice</u> is a finitely generated free R-module. By an <u>RG-lattice</u> X, one means a finitely generated RG-module which is an R-lattice. We denote the category of all RG-lattices by $\mathrm{mod}^{o}RG$. We aim to study the modular representation theory of $G = (G,B,N,\underline{R},U)$ with respect to the p-modular system (K,R,F) where p is the characteristic of G. Some of our arguments here will not depend on the modular system (K,R,F), and so we will often use k to denote any field such that $\mathrm{char}\, k \nmid |H|$ and k is a splitting field for H.

Let $X \in \mathrm{mod}^{o}RG$. If $k = K$ or F, we write $kX := k \otimes_R X$; kX is then a kG-module. We identify FX with the quotient $\bar{X} = X/(\pi)X$, and if $x \in X$, we denote by $\bar{x}$ the image of x under the natural map $X \to \bar{X}$.

Definitions: 4.0.1 ([PL], §14)

(i) Let $M \in \operatorname{mod} KG$. By an R-form of M we mean an RG-lattice X such that $KX = M$.

(ii) Let Λ be a finite dimensional K-algebra. An R-order A of Λ is an R-algebra with 1 , which is an R-lattice such that $KA = K \otimes_R A \cong \Lambda$.

The existence of R-forms and R-orders for finite dimensional KG-modules and finite dimensional K-algebras, respectively, follows from the fact that R is a principal ideal domain (see [CRII], 16.15). As an example let $\Lambda = KG$, the group algebra of a finite group G over K , and elt $A = RG$, the group ring of G over R . Then A is an R-order in Λ . If $X \in \operatorname{mod}^{o} RG$, we regard X as a subset of KX by identifying $x \in X$ with $1_K \otimes x \in KX$, in this way X is an R-form in KX .

If $X_1 , X_2 \in \operatorname{mod} RG$ then $(X_1,X_2)_{RG}$ is an R-submodule of $(X_1,X_2)_R$. Since $(X_1,X_2)_R$ is an R-lattice and R is a principle ideal domain, it follows that $(X_1,X_2)_{RG}$ is also an R-lattice.

If $k \in \{K,F\}$ then there is a k-map

$$\psi_k : k \otimes_R (X_1,X_2)_{RG} \to (kX_1,kX_2)_{kG} ,$$

which takes $c \otimes f$ to $c(\mathrm{Id}_k \otimes f)$ $(c \in k , f \in (X_1,X_2)_{RG}$; Id_k denotes the identity map of k). ψ_k is injective, and if $k = K$, it is surjective K-map (see [PL], Lemma 14.5). The F-map ψ_F is not surjective in general.

Definitions 4.0.2 (Green [G2], §2):

(i) The pair $X_1,X_2 \in \operatorname{mod}^{o} RG$ is said to be <u>p-stable</u> if the F-map

ψ_F , defined above is surjective.

(ii) If $X \in \mathrm{mod}^{\circ}RG$ then X is said to be <u>p-endostable</u> if the pair X , X is p-stable.

Note that part (i) of the above definition is equivalent to the condition that the map $\phi_F : (X_1,X_2)_{RG} \to (FX_1,FX_2)_{FG}$, defined by $f \to \bar{f} = \mathrm{Id}_F \otimes f$ $(f \in (X_1,X_2)_{RG})$, should be surjective. Note also that ϕ_F has kernel $\pi(X_1,X_2)_{RG}$.

* * * * *

For every $J \subseteq \underline{R}$, let $Y_J = RG_J[U]$, where G_J is the parabolic subgroup of G corresponding to J (see §1.2). It is clear that Y_J is isomorphic to the induced RG_J-lattice $\mathrm{Ind}_U^{G_J}(R_U)$, where R_U denotes the trivial RU-lattice. If $J = \underline{R}$; we write $Y_{\underline{R}} = Y$. We will denote by kY_J the kG_J-module $kG_J[U]$, and if $k \in \{K,F\}$, we identify kY_J with $k \underset{R}{\otimes} Y_J$.

Consider $E(Y_J) = \mathrm{End}_{RG_J}(Y_J)$. Since the elements of N_J form a set of representatives of the (U,U)-double cosets in G_J (see 1.2.6), $E(Y_J)$ has an R-basis $\{A_n^J , n \in N_J\}$, where $A_n^J \in E(Y_J)$ $(n \in N_J)$ is given by:

$$A_n^J([U]) := [UnU] \quad .$$

Since Y_J is permutation RG_J-lattice, Y_J is p-endostable (Scott [LS]; see [PL], p.174). Therefore

$$F \underset{R}{\otimes} E(Y_J) \;(= E(Y_J)/\pi E(Y_J)) \cong E(FY_J) \;,$$

where $E(FY_J) = \mathrm{End}_{FG_J}(FY_J)$. Since $K \underset{R}{\otimes} E(Y_J) \cong E(KY_J)$, we may regard

$E(Y_J)$ as an R-order in the K-algebra $E(KY_J)$.

Notations: If $n \in N_J$, denote by $A^J_{n,k}$ the k-basis element of $E(kY_J)$ which corresponds to n . If $k \in \{K,F\}$, we identify $A^J_{n,k}$ with $Id_k \underset{R}{\otimes} A^J_n \in k \underset{R}{\otimes} E(Y_J)$ ($\cong E(kY_J)$) . If $J = \underline{R}$, we write

$A^{\underline{R}}_{n,k} = A_{n,k}$, for every $n \in N_J$.

The set $\{A^J_{n,k}$, $n \in N_J\}$ form a k-basis for the k-algebra $E(kY_J)$.

Remark 4.0.3 Since R is complete in the (π)-adic topology, $E(Y_J)$ is also complete in the $\pi E(Y_J)$-adic topology. Therefore any idempotent $e \in E(FY_J)$ ($\cong E(Y_J)/\pi E(Y_J)$) can be lifted to an idempotent $e^0 \in E(Y_J)$ (i.e. $\bar{e}^0 = Id_F \underset{R}{\otimes} e^0 = e$) (see [CRII], p.123).

§4.1 The R-order E(Y)

In this section we consider the R-order $E(Y)$, where $Y = Y_{\underline{R}}$. We saw in the previous section that $E(Y)$ has an R-basis $\{A_n$, $n \in N\}$. In order to describe the multiplication of the elements A_n in $E(Y)$ we need the following:

Structural equation for G : ([C2], 4.4)

Let $w_i \in \underline{R}$. There exist functions $f_i : U_i^* \to U_i^*$, $h_i : U_i^* \to H$ and $g_i : U_i^* \to U$, with f_i bijective such that

$$(w_i)u(w_i) = f_i(u)h_i(u)(w_i)g_i(u) \quad \text{for all} \quad u \in U_i^* .$$

Here $U_i^* = U_i \backslash \{1\}$.

The proof of the following proposition was given by H. Sawada [HS] for the k-algebra $E(kY)$.

<u>Proposition 4.1.2</u> ([HS], Prop. 2.6): The multiplication in $E(Y)$ is given as follows:

(i) $A_n A_h = A_{hn}$ and $A_h A_n = A_{nh}$ for all $h \in H$, $n \in N$.

(ii) Suppose that $n \in N$, with $nH = w \in W$, then

$$A_{(w_i)} A_n = \begin{cases} A_{n(w_i)} & \text{if } \ell(ww_i) > \ell(w) \\ |U_i| A_{n(w_i)} + \left(\sum_{x \in U_i^*} A_{(w_i)h_i(x)(w_i)} \right) A_n & \text{if } \ell(ww_i) < \ell(w) , \end{cases}$$

and

$$A_n A_{(w_i)} = \begin{cases} A_{(w_i)n} & \text{if } \ell(w_i w) > \ell(w) \\ |U_i| A_{(w_i)n} + A_n \left(\sum_{x \in U_i^*} A_{h_i(x)} \right) & \text{if } \ell(w_i w) < \ell(w) . \end{cases}$$

□

The next proposition follows from 4.1.2.

<u>Proposition 4.1.3</u> $E(Y)$ is generated as an R-algebra by $\{A_{(w_i)}, A_h \mid w_i \in \underline{R}, h \in H\}$. □

By our assumption on the p-modular system (K,R,F), both K and F are splitting fields for G and all its subgroups. In particular, both K and F are splitting fields for H, and so each of them contains all the c-th roots of 1, where c is the exponent of H (see [CRI], p.37). Let $(k^{\times})_c$ denote the set of c-th root of 1 in k, and let $\Gamma : (F^{\times})_c \to (K^{\times})_c$ be a fixed multiplicative isomorphism such that

4.1.4 $\quad \overline{\Gamma(x)} = x \quad$ for all $x \in (F^{\times})_c$.

Note that any c-th root of 1 in K, say α, lies in R, because

α is integral over R (since $\alpha^c = 1$) and R is integrally closed in its quotient field K.

<u>Definition 4.1.5</u> We write $\hat{H} = \mathrm{Hom}(H, k^\times)$, the set of all multiplicative characters of H in k. We also write $\tilde{H} = \mathrm{Hom}(H, K^\times)$ and $\bar{H} = \mathrm{Hom}(H, F^\times)$.

For every $\chi \in \bar{H}$, define $\chi^o \in \tilde{H}$ by $\chi^o(h) := \Gamma(\chi(h))$, for all $h \in H$. It is clear that the map $\chi \to \chi^o$ gives a 1-1-correspondence between $\bar{H}$ and $\tilde{H}$. By 4.1.4, the multiplicative character $\chi^o \in \tilde{H}$ is a lift for χ (i.e. $\bar{\chi}^o = \chi$, where $\bar{\chi}^o(h) = \overline{\chi^o(h)}$ for all $h \in H$, and " - " means reduction $\mathrm{mod}(\pi)$).

Since $B = UH$ and U is a normal p-subgroup of B, any character $\chi \in \hat{H}$ can be extended to a character $\chi_B \in \mathrm{Hom}(B, k^X)$ having U in its kernel.

<u>Definition 4.1.7</u> For every $\chi \in \hat{H} = \mathrm{Hom}(H, k^X)$, let

$$\beta_\chi = |H|^{-1} \sum_{h \in H} \chi(h^{-1}) h .$$

It is clear that β_χ is an idempotent in kH and that $1 = \sum_{\chi \in \hat{H}} \beta_\chi$ is an orthogonal primitive idempotent decomposition of 1 in kH. Since H normalizes U, every $h \in H$ commutes with $[U]$ and so $\beta_\chi [U] = [U] \beta_\chi$.

For $\chi \in \hat{H}$, let $kY_\chi = kG[U]\beta_\chi$. kY_χ is isomorphic as kG-module to the induced kG-module L_χ^G, where L_χ is a one-dimensional FB-module affording χ_B.

<u>Definition 4.1.8</u> For $\chi \in \hat{H}$, define $e_\chi = |H|^{-1} \sum_{h \in H} \chi(h^{-1}) A_{h,k}$.

It is clear that e_χ $(\chi \in \hat{H})$ is an idempotent in $E(kY)$ and that

4.1.9 $$1_{E(kY)} = \Sigma_{\chi \in \hat{H}} e_\chi$$

is an orthogonal idempotent decomposition of $1_{E(kY)}$ in $E(kY)$. Since $A_{h,k}([U]) = [U]h$ $(h \in H)$, we have $e_\chi(kY) = kY_\chi$. Therefore we have

<u>Proposition 4.1.10</u> ([CL], Lemma 3.3)

$$kY = \Sigma^{\oplus}_{\chi \in \hat{H}} kY_\chi .$$

□

By remark 4.0.3, for every $\chi \in \bar{H}$, the idempotent $e_\chi \in E(FY)$ can be lifted to an idempotent $e^o_\chi \in E(Y)$. In fact, since $\chi^o (\in \tilde{H})$ is a lift for χ, and since $\chi^o(h) \in R$ $(h \in H)$, we have

$$e^o_\chi = e_{\chi^o} = |H|^{-1} \sum_{h \in H} \chi^o(h^{-1}) A_h .$$

Therefore, we can lift the equation 4.1.9 (putting $k = F$) to give an orthogonal idempotent decomposition in $E(Y)$.

4.1.11 $$1_{E(Y)} = \Sigma_{\chi \in \bar{H}} e_{\chi^o} = \Sigma_{\chi \in \tilde{H}} e_\chi .$$

Hence we have

<u>Proposition 4.1.12</u> $Y = \Sigma^{\oplus}_{\chi \in \tilde{H}} Y_\chi$, where $Y_\chi = RG[U]\beta_\chi$, for all $\chi \in \tilde{H}$.

<u>Proof</u> Follows directly from 4.1.11 and the fact that $e_\chi(Y) = Y_\chi$ $(\chi \in \tilde{H})$.

□

The Weyl group W acts on H by conjugation. This action will induce a W-action on $\hat{H} = \mathrm{Hom}(H,k^X)$ as follows: If $w \in W$ and $\chi \in \hat{H}$, then $w\chi \in \hat{H}$ is given by

4.1.13 $\quad w\chi(h) := \chi(h^{(w)})$ for all $h \in H$.

Now let $n \in N$ and suppose that $nH = w \in W$. Let $\chi \in \hat{H}$ and consider the element $A_n([U]\beta_\chi)$ of Y:

$$\begin{aligned}
A_n([U]\beta_\chi) &= A_n(\beta_\chi[U]) \\
&= \beta_\chi A_n([U]) \\
&= \beta_\chi[UnU] \\
&= \beta_\chi[U^-_{w^{-1}}nU] && \text{by 1.2.3} \\
&= \beta_\chi[U^-_{w^{-1}}]\; n\; [U] \\
&= [U^-_{w^{-1}}]\beta_\chi\; n\; [U] && \text{since } H \text{ normalizes } U^-_{w^{-1}} \\
&= [U^-_{w^{-1}}]\; |H|^{-1} \sum_{h\in H} \chi(h^{-1})hn[U] \\
&= [U^-_{w^{-1}}]\; |H|^{-1} \sum_{h\in H} \chi(h^{-1})nn^{-1}hn[U] \\
&= [U^-_{w^{-1}}]\; |H|^{-1} \sum_{h\in H} \chi({}^n(h^{-1}))nh[U] \\
&= [U^-_{w^{-1}}]\; n\; \beta_{w^{-1}\chi}\,[U] && \text{by 4.1.13} \\
&= [U^-_{w^{-1}}]\; n\; [U]\beta_{w^{-1}\chi}\;.
\end{aligned}$$

Therefore we have

<u>Lemma 4.1.14</u> ([CL], Lemma 3.14): Suppose $n \in N$ and let $nH = w \in W$.

Then A_n maps Y_χ into $Y_{w^{-1}\chi}$ for all $\chi \in \tilde{H}$.

Proof Calculation above gives

$$A_n([U]\beta_\chi) = [UnU]\beta_{w^{-1}\chi} \in Y_{w^{-1}\chi} \quad . \qquad \square$$

Notations: We shall use the following notations: If $\chi \in \hat{H} = \mathrm{Hom}(H,k^\times)$, we let $kY_\chi = e_\chi(kY)$. For $\chi \in \tilde{H} = \mathrm{Hom}(H,K^\times)$, we write $Y_\chi = e_\chi(Y)$, $KY_\chi = e_\chi(KY)$ and $FY_\chi = (FY)_{\bar{\chi}} = e_{\bar{\chi}}(FY)$, where $\bar{\chi}$ is the modular reduction of χ .

§4.2 The Simple FG-modules

The simple FG-modules have been studied by many authors. Curtis [C2] and Richen [FR] classified them in terms of the composition factors of the permutation module $FY = FG[U]$. Similar classification, but from a different point of view, was given by Carter and Lusztig [CL]. In this section, we outline the theory of Curtis and Richen, and mention some of the work of N. Tinberg [NT2] on the indecomposable FG-summands of FY . In the end we mention a theorem, due to Green [G3], which relates the simple FG-modules to the simple modules for the Hecke algebra E(FY).

Definition (see [CL],§5) For $\chi \in \hat{H}$, define $P(\chi) := \{w_i \in \underline{R} | \chi|_{H_i} = 1\}$.

The following lemma will be needed. For the proof see [CL].

Lemma 4.2.1 ([CL], Lemma 5.1) Suppose $\chi \in \hat{H}$. Then

$$w\chi = \chi \quad \text{for all} \quad w \in W_{P(\chi)} \quad . \qquad \square$$

Let $\chi \in \bar{H}$. In ([C2], Thm. 6.6), Curtis proved that χ can be extended uniquely to a character of $G_{P(\chi)}$ and $G_{P(\chi)}$ is the maximal parabolic subgroup of G with this property. If $J \subseteq P(\chi)$, we denote by $\rho_{\chi,J}$ the extension of χ to G_J . Let $L_{\chi,J}$ $(J \subseteq P(\chi))$ be a one-dimensional FG_J-module which affords $\rho_{\chi,J}$; we write $L_{\chi,\phi} = L_\chi$.

The proof of the following theorem was given in [C2] and [FR].

<u>Theorem 4.2.2</u> Let $\chi \in \bar{H}$, then

(i) $L_\chi^G \cong \sum\limits_{J\subseteq P(\chi)}^{\oplus} FY(\chi,J)$, where $FY(\chi,J)$ are indecomposable and mutually non-isomorphic.

(ii) If $S \subseteq P(\chi)$, then we have

$$L_{\chi,S}^G \cong \sum_{S\subseteq J\subseteq P(\chi)}^{\oplus} FY(\chi,J) \quad .$$

Moreover, the indecomposable FG-module $FY(\chi,J)$ have simple head and simple socle, and the socles of $FY(\chi,J)$ are precisely the simple FG-modules without repeat, as are the heads of the $FY(\chi,J)$. □

<u>Definition</u> If $J \subseteq \underline{R}$, $\chi \in \bar{H}$, we let $P_J(\chi) = P(\chi) \cap J$. The set $\{(\chi,S) \mid \chi \in \bar{H} , S \subseteq P_J(\chi)\}$ are called the admissible G_J-pairs.

Replacing G by G_J $(J \subseteq \underline{R})$, in theorem 4.2.2, we see that the indecomposable FG_J-summands of $L_\chi^{G_J}$ are indexed by the set $\{(\chi,S) \mid S \subseteq P_J(\chi)\}$ (see [NT2], Lemma 2.9). We denote the indecomposable FG_J-module corresponding to the admissible G_J-pair (χ,S) by $FY_J(\chi,S)$. It follows that

$$L_\chi^{G_J} \cong \sum_{S \subseteq P_J(\chi)}^{\oplus} FY_J(\chi,S) \ .$$

Since $(L_\chi^{G_J})^{G_J} \cong L_\chi^G$, and by theorem 4.2.2(i), the induced FG-module $(FY_J(\chi,S))^G$ is a direct sum of mutually non-isomorphic indecomposable FG-modules. The following lemma is due to N. Tinberg.

Lemma 4.2.3 ([NT2], Lemma 3.4) Let (χ,S) be an admissible G_J-pair for some $J \subseteq \underline{R}$, then

$$(FY_J(\chi,S))^G \cong \sum_{\substack{L \subseteq P(\chi) \\ L \cap J = S}} FY(\chi,L) \ . \qquad \square$$

Since $L_\chi^G \cong FY_\chi$, it follows from theorem 4.2.2(i) that FY_χ is a multiplicity free FG-module. Consequently FY ($\cong \sum_{\chi \ H}^{\oplus} FY_\chi$) is also a multiplicity free FG-module. Therefore the simple E(FY)-modules are all one-dimensional. The one-dimensional characters of E(FY) were determined by H. Sawada [HS]. If $\psi(\chi,J)$ denotes the one-dimensional character of E(FY) corresponding to the indecomposable FG-summand $FY(\chi,J)$ of FY , then $\psi(\chi,J)$ is given by

$$\psi(\chi,J)(A_{h,F}) = \chi(h) \qquad \text{for all } h \in H \ ,$$

4.2.4 and

$$\psi(\chi,J)(A_{(w_i),F}) = \begin{cases} -1 & \text{if } w_i \in P(\chi)\backslash J \\ 0 & \text{if } w_i \notin P(\chi)\backslash J \end{cases}$$

for all $w_i \in \underline{R}$. The set

$$\{\psi(\chi,J) \mid (\chi,J) \text{ is admissible G-pair}\}$$

is a complete set of the one-dimensional characters of $E(FY)$.

Let $M \in \text{mod } FG$. We saw in (§2.2, Prop. 2.2.15) that $I_U(M) := \{m \in M \mid um = m \ , \ \forall u \in U\}$ has a structure of right $E(FY)$-module. For an admissible G-pair (χ,J) , let $M(\chi,J)$ denote the head of the indecomposable FG-module $FY(\chi,J)$. The following theorem has been proved, in a more general setting, by Green [G2]. It gives a 1-1 correspondence between the simple FG-modules and the one-dimensional characters of $E(FY)$.

Theorem 4.2.5 (Green [G2], Theorem 2) The map $M \to I_U(M)$ induces a bijection between the set of simple FG-modules and the set of simple right $E(FY)$-modules. If $M = M(\chi,J)$ then $I_U(M)$ is the one-dimensional $E(FY)$-module affording the character $\psi(\chi,J)$. □

CHAPTER 5. The R-order S_χ

The R-lattice $Y = RG[U]$ (= $\sum^{\oplus}_{\chi \in \tilde{H}} Y_\chi$) is naturally left $E(Y)$-module by setting

$$Ay := A(y) \quad \text{for all} \quad A \in E(Y) \ , \quad y \in Y \ .$$

We saw in lemma 4.1.14 that the map A_n $(n \in N)$ of $E(Y)$ maps Y_χ into $Y_{w^{-1}\chi}$ for every $\chi \in \tilde{H}$, where $nH = w \in W$.

Definition 5.0.1

(i) For $\chi \in \hat{H} = \mathrm{Hom}(H, k^\times)$, denote by (χ) the W-orbit of χ, i.e. $(\chi) := \{w\chi \mid w \in W\}$.

Let $e_{(\chi)} := \sum_{\lambda \in (\chi)} e_\lambda$ and $kY_{(\chi)} := e_{(\chi)}(kY)$.

(ii) For $\chi \in \tilde{H} = \mathrm{Hom}(H, K^\times)$, defined $Y_{(\chi)} = e_{(\chi)}(Y)$.

Now let $\chi \in \tilde{H}$. By lemma 4.1.14, the RG-lattice $Y_{(\chi)}$ (= $\sum_{\lambda \in (\chi)} Y_\lambda$) is a left $(E(Y), RG)$-bimodule. It is clear that $e_{(\chi)}$ is an idempotent in $E(Y)$. Let $S_\chi = e_{(\chi)}E(Y)e_{(\chi)}$. Since $Y_{(\chi)} = e_{(\chi)}(Y)$, it follows (see appendix, lemma 1) that S_χ is an R-algebra isomorphic to $E(Y_{(\chi)})$.

Notations 5.0.2 For $\chi \in \hat{H} = \mathrm{Hom}(H, k^\times)$, we let $kS_\chi := e_{(\chi)}E(kY)e_{(\chi)}$, and if $\chi \in \tilde{H} = \mathrm{Hom}(H, K^\times)$, we write FS_χ for the F-algebra $e_{(\bar{\chi})}E(FY)e_{(\bar{\chi})}$. It is clear that if $\chi \in \mathrm{Hom}(H, k^\times)$ then $e_{(\chi)}$ is an idempotent in $E(kY)$ and $kS_\chi \cong E(e_{(\chi)}(kY))$.

The RG-lattice $Y_{(\chi)}$ $(\chi \in \hat{H})$ is p-endostable, since it is a direct RG-summand of the p-endostable RG-lattice Y (see [G2], Lemma 2.4(i)). Therefore, we have $F \otimes_R E(Y_{(\chi)}) = E(FY_{(\chi)})$ and so

$$FS_\chi \cong E(FY_{(\chi)}) \cong E(Y_{(\chi)})/\pi E(Y_{(\chi)}) \cong S_\chi/\pi S_\chi \ .$$

Hence we may regard S_χ as an R-order in KS_χ .

In this chapter, we consider the system $(KS_\chi, S_\chi, FS_\chi)$. The significance of studying this system is given by the following:

Proposition 5.0.3

(i) If $\lambda,\mu \in \hat{H}$, then $\mathrm{Hom}_{RG}(Y_\lambda, Y_\mu) = 0$ unless $\mu \in (\lambda)$.

(ii) $E(Y) \cong \coprod S_\chi$, where the sum is taken over a set of representatives of the W-orbits of $\hat{H} = \mathrm{Hom}(H, K^X)$.

Proof (i) If $\lambda \in \hat{H}$, let R_λ denote the RB-lattice of rank one which affords λ . It is clear that $Y_\lambda \cong R_\lambda^G$, for every $\lambda \in \hat{H}$. From Mackey decomposition and since the elements of W form a set of (B,B)-double coset representatives of G , we have

$$\begin{aligned}(Y_\lambda, Y_\mu)_{RG} &\cong (R_\lambda^G, R_\mu^G)_{RG}\\ &\cong \sum_{w \in W} (R_{w\lambda}, R_\mu)_{B \cap {}^wB} \ .\end{aligned}$$

Since $H \le B \cap {}^wB$, for all $w \in W$, we then have $(R_{w\lambda}, R_\mu)_{B\cap {}^wB} = 0$ unless $w\lambda = \mu$; i.e. unless $\mu \in (\lambda)$.

(ii) Follows from (i) and the fact that $S_\chi \cong E(Y_{(\chi)})$.

□

It is clear that S_χ , $\chi \in \hat{H}$, is the R-span of the set

$$\{e_\lambda A_n e_\mu \mid \lambda,\mu \in (\chi) \text{ and } n \in N\} \ .$$

But since $e_\lambda A_h = \lambda(h)e_\lambda$, for all $h \in H$ and all $\lambda \in (\chi)$, it follows from 4.1.2(i) that S_χ is the R-span of the set

$$\{e_\lambda A_{(w)} e_\mu \mid \lambda,\mu \in (\chi) \text{ and } w \in W\} \ .$$

Let $\lambda,\mu \in \hat{H}$, and let $w \in W$. Consider $e_\lambda A_{(w)} e_\mu \in E(Y)$,

$$e_\lambda A_{(w)} e_\mu([U]) = \sum_{h,h'} \frac{1}{|H|^2} \lambda(h^{-1})\mu(h'^{-1})A_h A_{(w)} A_{h'}([U])$$

$$= \sum_{h,h'} \frac{1}{|H|^2} \lambda(h^{-1})\mu(h'^{-1})h'[U^-_{w^{-1}}](w)h[U] \quad \text{by 1.2.3}$$

$$= \sum_{h,h'} \frac{1}{|H|^2} \lambda(h^{-1})\mu(h'^{-1})h'[U^-_{w^{-1}}](w)h(w)^{-1}(w)[U]$$

$$= \sum_{h,h'} \frac{1}{|H|^2} \lambda(h^{-1})\mu(h'^{-1})h'[U^-_{w^{-1}}]^{(w)}h(w)[U]$$

$$= \sum_{h,h'} \frac{1}{|H|^2} \lambda(h^{-1})\mu(h'^{-1})h'^{(w)}h[U^-_{w^{-1}}](w)[U]$$

since H normalizes $U^-_{w^{-1}}$

5.0.4 $$= \beta_\mu \beta_{w\lambda} [U^-_{w^{-1}}](w)[U] \ .$$

Hence we have

Lemma 5.0.5 Suppose $\lambda,\mu \in \tilde{H}$ and $w \in W$, then

$$e_\lambda A_{(w)} e_\mu = \begin{cases} A_{(w)} e_\mu & \text{if } w\lambda = \mu \\ 0 & \text{if } w\lambda \neq \mu \end{cases} .$$

Proof From 5.0.4 we have

$$\begin{aligned} e_\lambda A_{(w)} e_\mu\ ([U]) &= \beta_\mu \beta_{w\lambda}([U(w)U] \\ &= \begin{cases} \beta_\mu [U(w)U] & \text{if } w\lambda = \mu \\ 0 & \text{if } w\lambda \neq \mu \end{cases} . \end{aligned}$$

But $$\begin{aligned} \beta_\mu [U(w)U] &= \frac{1}{|H|} \Sigma\ \mu(h^{-1}) h [U^-_{w^{-1}}](w)[U] \\ &= A_{(w)} e_\mu\ ([U]) . \end{aligned}$$

□

Lemma 5.0.6 $A_{(w)} e_\mu = e_{w^{-1}\mu} A_{(w)}$, for all $\mu \in \tilde{H}$ and all $w \in W$.

Proof $$\begin{aligned} e_{w^{-1}\mu} A_{(w)} &= e_{w^{-1}\mu} A_{(w)} 1_{E(Y)} \\ &= e_{w^{-1}\mu} A_{(w)} \big(\Sigma_{\lambda\in\tilde{H}} e_\lambda \big) \\ &= \Sigma_{\lambda\in\tilde{H}}\ e_{w^{-1}\mu} A_{(w)} e_\lambda . \end{aligned}$$

By 5.0.5, $e_{w^{-1}\mu} A_{(w)} e_\lambda = 0$ unless $\lambda = \mu$, and $e_{w^{-1}\mu} A_{(w)} e_\mu = A_{(w)} e_\mu$.
Therefore $e_{w^{-1}\mu} A_{(w)} = A_{(w)} e_\mu$. □

We now describe an R-basis for the R-order S_χ .

<u>Proposition 5.0.7</u> For $\chi \in \hat{H}$, the R-order S_χ is of rank $|W|\,|(\chi)|$. In fact the set $\{e_\lambda A_{(w)} \mid \lambda \in (\chi)$ and $w \in W\}$ is an R-basis for S_χ.

<u>Proof</u> Since S_χ is the R-span of $\{e_\lambda A_{(w)}e_\mu \mid \lambda,\mu \in (\chi)$ and $w \in W\}$, it follows from 5.0.5 and 5.0.6 that S_χ is the R-span of the set $T = \{e_\lambda A_{(w)} \mid \lambda \in (\chi)$ and $w \in W\}$. It remains to show that the elements of T are linearly independent. Suppose that

5.0.8 $$\sum_{\substack{\lambda\in(\chi)\\ w\in W}} c_{\lambda,w}\, e_\lambda A_{(w)} = 0 ,$$

where $c_{\lambda,w} \in R$, for all $\lambda \in (\chi)$ and all $w \in W$. Then by multiplying the equation 5.0.8 from the left by e_μ $(\mu \in (\chi))$, and since $e_\mu e_\lambda = 0$, for every $\lambda \in (\chi)$ with $\lambda \neq \mu$, we then have

5.0.9 $$\sum_{w\in W} c_{\mu,w}\, e_\mu A_{(w)} = 0 .$$

But since $e_\mu A_{(w)}$ $(w \in W)$ is a linear combination of elements A_n, with $n \in (w)H$, and since the elements A_n $(n \in N)$ are linearly independent in $E(Y)$, equation 5.0.9 implies that $c_{\mu,w} = 0$ for all $w \in W$. Repeating this process for all $\mu \in (\chi)$, we get $c_{\mu,w} = 0$ for all $w \in W$ and all $\mu \in (\chi)$. This proves that the elements of T are linearly independent and hence completes the proof of 5.0.7. □

The next proposition describes the multiplication rule in S_χ.

<u>Proposition 5.0.10</u> Let $\chi \in \hat{H}$.

(i) For $\lambda,\mu \in (\chi)$, we have

$$e_\lambda A_{(w)} e_\mu A_{(v)} \neq 0 \iff \mu = w\lambda , \text{ for all } v,w \in W .$$

(ii) If $w \in W$, $w_i \in \underline{R}$ and $\lambda \in (\chi)$, then

$$e_\lambda A_{(w)} e_{w\lambda}\ A_{(w_i)} = \begin{cases} e_\lambda A_{(w_i)(w)} & \text{if } \ell(w_i w) > \ell(w) \\ |U_i| e_\lambda A_{(w_i)(w)} + (w\lambda)_i e_\lambda A_{(w)} & \text{if } \ell(w_i w) < \ell(w) \end{cases}$$

and

$$e_\lambda A_{(w_i)} e_{w_i\lambda} A_{(w)} = \begin{cases} e_\lambda A_{(w)(w_i)} & \text{if } \ell(w w_i) > \ell(w) \\ |U_i| e_\lambda A_{(w)(w_i)} + (w_i\lambda)_i e_\lambda A_{(w)} & \text{if } \ell(w w_i) < \ell(w) , \end{cases}$$

where, for any $\mu \in (\chi)$ and $w_i \in \underline{R}$,

$$\mu_i := \sum_{x \in U_i^*} \mu(h_i(x)) .$$

<u>Proof</u> (i) Follows easily from 5.0.6 and the fact that $e_\lambda e_\mu = 0$, for all $\lambda,\mu \in (\chi)$ with $\lambda \neq \mu$.

(ii) From 5.0.6 and 4.1.2, we have

$$e_\lambda A_{(w)} e_{w\lambda}\ A_{(w_i)} = e_\lambda\ A_{(w)} A_{(w_i)}$$

$$= e_\lambda \begin{cases} A_{(w_i)(w)} & \text{if } \ell(w_i w) > \ell(w) \\ |U_i| A_{(w_i)(w)} + A_{(w)} \left(\sum_{x \in U_i^*} A_{h_i(x)} \right) & \text{if } \ell(w_i w) < \ell(w) \end{cases}$$

$$= \begin{cases} e_\lambda\ A_{(w_i)(w)} & \text{if } \ell(w_i w) > \ell(w) \\ |U_i| e_\lambda\ A_{(w_i)(w)} + A_{(w)} e_{w\lambda} \left(\sum_{x \in U_i^*} A_{h_i(x)} \right) & \text{if } \ell(w_i w) < \ell(w) \end{cases}$$

$$= \begin{cases} e_\lambda\, A_{(w_i)(w)} & \text{if } \ell(w_i w) > \ell(w) \\ |U_i| e_\lambda A_{(w_i)(w)} + (\sum_{x \in U_i^*} w\lambda(h_i(x)))e_\lambda A_{(w)} & \text{if } \ell(w_i w) < \ell(w) \end{cases} .$$

The proof of the second part of (ii) is similar. □

The next Corollary follows from 5.0.10.

<u>Corollary 5.0.11</u> S_χ , $\chi \in \hat{H}$, is generated as an R-algebra by $\{e_\lambda A_{(w_i)} \mid \lambda \in (\chi) \text{ and } w_i \in \underline{R}\}$. □

<u>Remark 5.0.12</u> Replacing $Y_{(\chi)}$ and S_χ $(\chi \in \hat{H})$ by $kY_{(\chi)}$ and kS_χ $(\chi \in \hat{H})$, respectively, it is easy to see that all the previous results for S_χ are valid for kS_χ . In particular kS_χ is of dimension $|W|\ |(\chi)|$, in fact kS_χ has a k-basis $\{e_\lambda A_{(w),k} \mid \lambda \in (\chi) , w \in W\}$. If $k \in \{K,F\}$, we identify $e_\lambda A_{(w),k}$ with $Id_k \otimes e_\lambda A_{(w)}$ $(\lambda \in (\chi), w \in W)$.

The following lemma, concerning the coefficient $\mu_i = \sum_{x \in U_i^*} \mu(h_i(x))$ $(\mu \in \hat{H})$, was proved by Carter and Lusztig [CL]for an arbitrary field k .

<u>Lemma 5.0.13</u> ([CL], Cor. 3.12) Suppose that $\chi \in \hat{H}$. If $w_i\chi \neq \chi$, $w_i \in \underline{R}$, then $\sum_{x \in U_i^*} \chi(h_i(x)) = 0$. □

<u>Proposition 5.0.14</u> Suppose that char $k \nmid p$, and let $\chi \in \hat{H}$. Then

(i) The map $\theta : kY_\lambda \to kY_{w_i\lambda}$, induced by $e_{w_i\lambda}\, A_{(w_i),k}$, is a kG-isomorphism, for all $w_i \in \underline{R}$ and all $\lambda \in (\chi)$.

(ii) $kY_\lambda \cong kY_\mu$ as kG-modules, for all $\lambda,\mu \in (\chi)$.

<u>Proof</u> (i) We may assume that $w_i\lambda \neq \lambda$. Let $\beta : kY_{w_i\lambda} \to kY_\lambda$ be the kG-map induced by $e_\lambda A_{(w_i),k}$. Consider $\beta\theta \in E(kY_\lambda)$. $\beta\theta$ is induced by

$$e_\lambda A_{(w_i),k} e_{w_i\lambda} A_{(w_i),k} = e_\lambda A^2_{(w_i),k} \qquad \text{by 5.0.6}$$

$$= e_\lambda(|U_i|\ A_{(w_i)^2,k} + (\sum_{x\in U_i^*} A_{(w_i)^{-1}h_i(x)(w_i)})A_{(w_i),k} \quad , \quad \text{by 5.0.10}$$

$$= |U_i|\lambda((w_i)^2)e_\lambda + (\sum_{x\in U_i^*} w_i\lambda(h_i(x)))\ e_\lambda A_{(w_i),k} \ , \quad \text{since } (w_i)^2 \in H$$

$$= |U_i|\lambda((w_i)^2)e_\lambda \qquad \text{by 5.0.13, since } w_i\lambda \neq \lambda$$

$$= ce_\lambda \quad , \quad \text{where } c = |U_i|\lambda((w_i)^2) \in k \ .$$

Similarly $\theta\beta \in E(kY_{w_i\lambda})$ is induced by $c'e_{w_i\lambda}$, where $c' = |U_i|\ w_i\lambda((w_i)^2) \in k$. Since $|U_i|$, $\lambda((w_i)^2)$ and $w_i\lambda((w_i)^2)$ are all non-zero in k , c and c' are non-zero in k and so we have $c^{-1}\beta\theta = e_\lambda$ and $\theta.c'^{-1}\beta = e_{w_i\lambda}$. Therefore λ is a k-isomorphism and hence kG-isomorphism, since it is a kG-map.

(ii) If $\lambda,\mu \in (\chi)$ then $\mu = w\lambda$ for some $w \in W$. Let $w = w_{i_1} w_{i_2} \dots w_{i_m}$ be a reduced expression of w . Then, by (i), we have a sequence

$$kY_\mu \to kY_{w_{i_2}\dots w_{i_m}\lambda} \to \dots \to kY_{w_{i_m}\lambda} \to kY_\lambda$$

of kG-isomorphisms. Therefore $kY_\mu \cong kY_\lambda$ as kG-modules.

$\square$

Since the multiplication relations in 5.0.10 involve $|U_i|$, and since $|U_i|$ is a power of p ([C2], 3.3) (hence $|U_i| \in \pi R$), the multiplication rule in FS_χ will have a simpler form. In fact we have

<u>Lemma 5.0.15</u> Let $\chi \in \hat{H}$.

(i) For $\lambda,\mu \in (\chi)$, we have

$$e_\lambda A_{(w),F}\, e_\mu\, A_{(v),F} \neq 0 \iff \mu = w\lambda \qquad \forall \quad v,w \in W .$$

(ii) If $w \in W$, $w_i \in \underline{R}$, and $\lambda \in (\chi)$, then

$$e_\lambda A_{(w),F} e_{w\lambda} A_{(w_i),F} = \begin{cases} e_\lambda A_{(w_i)(w),F} & \text{if } \ell(w_i w) > \ell(w) \\ (w\lambda)_i e_\lambda A_{(w_i)(w),F} & \text{if } \ell(w_i w) < \ell(w) , \end{cases}$$

and

$$e_\lambda A_{(w_i),F} e_{w_i\lambda} A_{(w),F} = \begin{cases} e_\lambda\, A_{(w)(w_i),F} & \text{if } \ell(ww_i) > \ell(w) \\ (w_i\lambda)_i e_\lambda A_{(w)(w_i)} & \text{if } \ell(ww_i) < \ell(w) . \end{cases}$$

□

* * * * *

§5.1 <u>The case when</u> χ <u>is regular</u>

<u>Definition</u> Let $\chi \in \hat{H}$. χ is said to be <u>regular</u> if $w\chi = \chi$, $w \in W$, implies that $w = 1$.

In this section we discuss the case when $\chi \in \hat{H}$ is regular. First we have

<u>Proposition 5.1.1</u> Suppose $\chi \in \hat{H}$ is regular. Then

(i) The kG-module kY_χ is indecomposable.

(ii) If $\chi \in \hat{H}$ then the RG-lattice Y_χ is indecomposable.

(iii) If $k = K$, then KY_χ is absolutely irreducible.

<u>Proof</u> (i) We have $kY_\chi = kG[U]\beta_\chi = e_\chi(kY)$, and e_χ is an idempotent in $E(kY)$. Consider the endomorphism algebra $E(kY_\chi)$. We have $E(kY_\chi) \cong e_\chi E(kY)e_\chi$ (see appendix, lemma 1), and the k-algebra $e_\chi E(kY)e_\chi$ is the k-span of $\{e_\chi A_{(w),k}\, e_\chi \mid w \in W\}$. But since χ is regular, it follows from 5.0.5 that $e_\chi A_{(w),k}\, e_\chi = 0$ for all $1 \neq w \in W$. Therefore $e_\chi E(kY)e_\chi = k.e_\chi$, and so $E(kY_\chi)$ is a local ring (see [CRII], Prop. 5.21) which implies that kY_χ is indecomposable kG-module.

(ii) Follows from case $k = F$ of (i).

(iii) From the proof of (i), we have $E(KY_\chi) \cong K$. But KY_χ is completely reducible, hence KY_χ is absolutely irreducible.

<u>Remark 5.1.2</u> If $\chi \in \hat{H}$ then, from 5.0.14(ii), we have $KY_\lambda \cong KY_\mu$ as KG-modules for all $\lambda,\mu \in (\chi)$. If χ is regular, it follows from 5.1.1(iii) that KY_λ is absolutely irreducible for all $\lambda \in (\chi)$. Consequently $(KY_\lambda, KY_\mu) \cong E(KY_\lambda) \cong K$ (Schur's Lemma) for all $\lambda,\mu \in (\chi)$. Therefore the K-algebra KS_χ ($\cong E(KY_{(\chi)})$) (which is semisimple in this case) is isomorphic to a full matrix algebra $M_{|W|}(K)$.

<u>The F-algebra FS_χ</u>

We are still assuming that $\chi \in \hat{H}$ is regular. We will consider the F-algebra $FS_\chi = F \otimes_R S_\chi$. It follows from 5.0.13 that, if $\mu \in (\chi)$, then

$\sum_{x \in U_i^*} \mu(h_i(x)) = 0$ for all $w_i \in \underline{R}$.

The next lemma follows from 5.0.15.

Lemma 5.1.3 In the F-algebra FS_χ ,

$$e_\lambda A_{(w),F}\, e_{w\lambda}\, A_{(v),F} = \begin{cases} e_\lambda\, A_{(v)(w),F} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{if } \ell(vw) \neq \ell(v) + \ell(w) \end{cases},$$

for all $\lambda \in (\chi)$ and all $w,v \in W$. □

Since $e_\lambda(FY) = FY_\lambda$ is an indecomposable FG-module (5.1.1(i)) for all $\lambda \in (\chi)$, it follows that e_λ is primitive for all $\lambda \in (\chi)$ and $e_{(\chi)} = \sum_{\lambda \in (\chi)} e_\lambda$ is an orthogonal primitive idempotent decomposition of $e_{(\chi)}$ $(= 1_{FS_\chi})$ in FS_χ . Therefore

5.1.4 $$FS_\chi = \sum_{\lambda \in (\chi)}^{\oplus} e_\lambda FS_\chi$$

is a decomposition of FS_χ into a direct sum of projective indecomposable right FS_χ-modules. We shall prove below (see proof of 5.1.5) that the F-algebra FS_χ has $|W|$ $(= |(\chi)|)$ simple modules.

For each $0 \le r \le n_0 = |\Phi^+|$, define J_r to be the subspace of FS_χ spanned by the set $\{e_\lambda A_{(w),F} \mid \lambda \in (\chi)\ ,\ w \in W \text{ with } \ell(w) \ge r\}$.

Proposition 5.1.5

(i) J_r is 2-sided ideal of FS_χ for all $1 \le r \le n_0$.

(ii) $J_1^r = J_r$ for all $1 \le r \le n_0$.

(iii) $J_1 = \underline{r}(FS_\chi)$, the Jacobson radical of FS_χ .

(iv) $FS_\chi > J_1 > J_2 > \ldots > J_{n_0} > 0$ is the radical series of FS_χ .

<u>Proof</u> (i) Clear from 5.1.3.

(ii) We use the induction on r . It is obviously true for $r = 1$. So let $1 < n \leq n_0$ and suppose that $J_1^n = J_n$. Consider J_1^{n+1} ; $J_1^{n+1} = J_1^n J_1 = J_n J_1$ is the F-span of the set

$$\{e_\lambda A_{(w),F}\, e_\mu\, A_{(v),F} \mid \lambda,\mu \in (\chi) \text{ , and } v,w \in W \text{ with } \ell(w) \geq n \text{ and } \ell(v) \geq 1\} .$$

By 5.0.10 and 5.1.3, if $\lambda,\mu \in (\chi)$ and if $v,w \in W$ then $e_\lambda\, A_{(w),F}\, e_\mu\, A_{(v),F} = 0$ if $\mu \neq \lambda$ and

$$e_\lambda\, A_{(w),F} e_{w\lambda}\, A_{(v),F} = e_\lambda\, A_{(w),F}\, A_{(v),F}$$

$$= \begin{cases} e_\lambda A_{(v)(w),F} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{if } \ell(vw) \neq \ell(v) + \ell(w) . \end{cases}$$

If $(v)(w) = (vw)h$ for some $h \in H$, and $\ell(vw) = \ell(v) + \ell(w)$ then $\ell(vw) \geq n + 1$ and

$$\begin{aligned} e_\lambda A_{(v)(w),F} &= e_\lambda A_{(vw)h,F} \\ &= e_\lambda A_{h,F}\, A_{(vw),F} \qquad \text{by 5.0.15(ii)} \\ &= \lambda(h)\, e_\lambda\, A_{(vw),F} \in J_{n+1} . \end{aligned}$$

Therefore $J_n J_1 \subseteq J_{n+1}$. Conversely, if $w \in W$ with $\ell(w) = n+1$, then we can write $w = w_i v$ with $v^{-1}(\alpha_i) > 0$, where α_i is the simple root corresponds to w_i . It follows that $\ell(v) \geq n$ and if $\lambda \in (\chi)$ then

$$\begin{aligned}
e_\lambda A_{(w),F} &= e_\lambda A_{(w_i v),F} \\
&= e_\lambda A_{(w_i)(v)h',F} \qquad \text{for some } h' \in H \\
&= e_\lambda A_{h',F} A_{(w_i)(v),F} \\
&= \lambda(h')\, e_\lambda A_{(v),F} A_{(w_i),F} \qquad \text{by 5.0.15(ii)} \\
&= \lambda(h')\, e_\lambda A_{(v),F} e_{v\lambda} A_{(w_i),F} \quad \text{by 5.0.15(i)} \\
&\in J_n J_1 \quad .
\end{aligned}$$

Hence $J_{n+1} \subseteq J_n J_1$ and so $J_1^{n+1} = J_n J_1 = J_{n+1}$, which completes the induction and hence the proof of (ii).

(iii) It is clear from (ii) that J_1 is nilpotent 2-sided ideal, hence $J_1 \subseteq \underline{r}(FS_\chi)$ ([CRII], Prop. 5.15). To show that $\underline{r}(FS_\chi) \subseteq J_1$, it is enough to show that FS_χ/J_1 is semisimple (see [CRII], Cor. 5.2). It is clear that $FS_\chi/J_1 = \sum^{\oplus}_{\lambda\in(\chi)} F.(e_\lambda + J_1)$, and that for all $\lambda \in (\chi)$,

$$(e_\lambda + J_1).e_\mu = \begin{cases} e_\lambda + J_1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

and

$$(e_\lambda + J_1)e_\mu A_{(w_i),F} = 0 \qquad \text{for all } \mu \in (\chi)\ ,\ w_i \in \underline{R}\ .$$

Therefore, putting $S_\lambda := F.(e_\lambda + J_1)$, S_λ is a one-dimensional (hence simple) FS_χ-module affording the character

$$\begin{aligned}\psi_\lambda : e_\lambda &\to 1 \\ : e_\mu &\to 0 \qquad\qquad \text{for all } \mu \neq \lambda \\ : e_\mu A_{(w_i),F} &\to 0 \qquad\qquad \text{for all } \mu \in (\chi) \text{ and all } w_i \in \underline{R} .\end{aligned}$$

It is clear that if $\lambda,\mu \in (\chi)$ then $\psi_\lambda \neq \psi_\mu$, if $\lambda \neq \mu$. Hence FS_χ/J_1 is semisimple right FS_χ-module and so $\underline{r}(FS_\chi) \subseteq J_1$. Therefore $J_1 = \underline{r}(FS_\chi)$.

(iv) It is enough to show that J_r/J_{r+1} is semisimple right FS_χ-module or equivalently $J_r\underline{r}(FS_\chi)$ $(= J_rJ_1$, by (iii)) $= J_{r+1}$. But this follows from (iii). This completes the proof of 4.3.20. □

Note that since $FS_\chi/J_1 \cong \sum^{\oplus}_{\lambda\in(\chi)} S_\lambda$, $\{S_\lambda, \lambda \in (\chi)\}$ form a full set of simple right FS_χ-modules.

As a corollary to the proof of 5.1.5(ii) we have

<u>Lemma 5.1.6</u> For every $1 \leq r \leq n_0$, J_1^r is generated as right FS_χ-module by the set

$$\{e_\lambda A_{(w),F} / \lambda \in (\chi) , w \in W \text{ with } \ell(w) = r\} .$$ □

For every $\lambda \in (\chi)$, let $P_\lambda = e_\lambda FS_\chi$. From 5.1.4, P_λ is a projective indecomposable right FS_χ-module for all $\lambda \in (\chi)$, and

$$FS_\chi = \sum^{\oplus}_{\lambda\in(\chi)} P_\lambda .$$

It is clear that $\dim_F P_\lambda = |W|$ for all $\lambda \in (\chi)$; in fact $P_\lambda = \sum^{\oplus}_{w\in W} F.e_\lambda A_{(w),F}$. By 5.1.5(iv), P_λ has a Loewy series ([PL], p.27)

5.1.7 $\qquad X_0 = P_\lambda > X_1 > \dots > X_{n_0} > 0$,

where, for all $0 \le r \le n_0$,

$$X_r = P_\lambda J_1^r = \sum_{\substack{w \in W \\ \ell(w) \ge r}}^{\oplus} F.e_\lambda A_{(w),F} \;.$$

Fix $\lambda \in (\chi)$, and consider the semisimple quotient X_r/X_{r+1} , $0 \le r \le n_0 - 1$, in the Loewy series 5.1.7 of P_λ .

$$X_r/X_{r+1} = \sum_{\substack{w \in W \\ \ell(w) = r}}^{\oplus} F.(e_\lambda A_{(w),F} + X_{r+1}) \;.$$

If $\mu \in (\chi)$, and $w_i \in \underline{R}$, then by 5.0.5

(*) $$(e_\lambda A_{(w),F} + X_{r+1})e_\mu = e_\lambda A_{(w),F}e_\mu + X_{r+1}$$

$$= \begin{cases} 0 & \text{if } \mu \ne w\lambda \\ e_\lambda A_{(w),F} + X_{r+1} & \text{if } \mu = w\lambda \;, \end{cases}$$

and

$$(e_\lambda A_{(w),F} + X_{r+1})e_\mu A_{(w_i),F} = e_\lambda A_{(w),F}e_\mu A_{(w_i),F} + X_{r+1}$$

$$= \begin{cases} 0 & \text{if } \mu \ne w\lambda \text{ or } \ell(w_i w) \ne \ell(w) + 1 \\ e_\lambda A_{(w_i)(w),F} + X_{r+1} & \text{if } \mu = w\lambda \text{ and } \ell(w_i w) = \ell(w) + 1 \;. \end{cases}$$

But $e_\lambda A_{(w_i)(w),F}$ is a scalar multiple of $e_\lambda A_{(w_i w),F} \in X_{r+1}$ (since $\ell(w) = r$) , therefore

(**) $\quad (e_\lambda A_{(w),F} + X_{r+1})e_\mu A_{(w_i),F} = 0$, for all $\mu \in (\chi)$, $w_i \in \underline{R}$.

From (*) and (**), it follows that $F.(e_\lambda A_{(w),F} + X_{r+1})$ is one-dimensional right FS_χ-module which affords the character $\psi_{w\lambda}$, hence $F.(e_\lambda A_{(w),F} + X_{r+1}) \cong S_{w\lambda}$.

Summarizing the above we have

<u>Proposition 5.1.8</u> Let $\chi \in \hat{H}$ be regular, and let $\lambda \in (\chi)$. Let P_λ be the projective indecomposable right FS_χ-module whose Loewy series is given by 5.1.7. Then

(i) For all $0 \le r \le \ell$, X_r is generated as an FS_χ-module by the set $\{e_\lambda A_{(w),F} \mid w \in W \text{ with } \ell(w) = r\}$.

(ii) $X_r/X_{r+1} \cong \sum^{\oplus}_{\substack{w\in W \\ \ell(w)=r}} S_{w\lambda}$, for all $0 \le r \le \ell-1$.

(iii) For all $w \in W$, $S_{w\lambda}$, appears as a composition factor of P_λ with multiplicity 1 .

<u>Proof</u> (i) Follows from the fact that $X_r = P_\lambda J_1^r$ and using 5.1.6.

(ii) Clear since $X_r/X_{r+1} = \sum^{\oplus}_{\substack{w\in W \\ \ell(w)=r}} F.(e_\lambda A_{(w),F} + X_{r+1})$, and $F.(e_\lambda A_{(w),F} + X_{r+1}) \cong S_{w\lambda}$.

(iii) Follows from (ii) and the fact that $\dim_F P_\lambda = |W|$. □

<u>Remark</u> It follows from 5.1.8 (iii) that the Cartan matrix C of the F-algebra FS_χ has the form

$$C = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \\ 1 & 1 & \dots & 1 \end{pmatrix}_{|W|\times|W|} .$$

In particular, C is singular unless $|W| = 1$.

*   *   *   *   *

Let $S = \sum_{\lambda\in(\chi)}^{\oplus} F.e_\lambda$. S is an F-subalgebra of FS_χ with identity element $e_{(\chi)}$. Let $\lambda \in (\chi)$, and consider the projective indecomposable right FS_χ-module P_λ as right S-module. For all $\mu \in (\chi)$, and all $w \in W$

$$e_\lambda A_{(w),F}\, e_\mu = \begin{cases} 0 & \text{if } \mu \neq w\lambda \\ e_\lambda A_{(w),F} & \text{if } \mu = w\lambda \end{cases} .$$

Therefore, for all $w \in W$, $F.e_\lambda A_{(w),F}$ is a one-dimensional right S-module affording the character

$$\begin{aligned} e_\mu &\to 0 \qquad \text{if } \mu \neq w\lambda , \\ e_{w\lambda} &\to 1 \qquad . \end{aligned}$$

Hence for every subset T of W , if we let $M_T = \sum_{w\in T}^{\oplus} F.e_\lambda A_{(w),F}$, then M_T is a right S-submodule of P_λ . Conversely, if M is an S-submodule of P_λ , then for all $\mu \in (\chi)$, $Me_\mu \neq 0 \iff \mu = w\lambda$ for some $w \in W$, in which case $Me_\mu = F.e_\lambda A_{(w),F} \leq M$, hence $M = \sum_{w\in T} F.e_\lambda A_{(w),F} = M_T$ for some subset T of W . It is clear from the above argument that, if T_1, T_2 are subsets of W , then $T_1 \subseteq T_2 \iff M_{T_1}$ is S-submodule of M_{T_2} . Therefore we have

<u>Proposition 5.1.9</u> The map $T \to M_T$ gives a lattice isomorphism between the lattice of subsets of W and the lattice of S-submodules of P_λ . □

Now, for a given subset T of W , we would like to know under which condition the S-submodule M_T of P_λ will be an FS_χ-submodule.

We need the following definition.

Definition 5.1.10

(i) If $v, w \in W$, write $v \overset{\cdot}{\leq} w$ if $w = w_i v$ for some $w_i \in \underline{R}$ such that $v^{-1}(\alpha_i) \in \Phi^+$, i.e. such that $\ell(w_i v) > \ell(v)$.

(ii) (Bruhat ordering): Say $v \leq w$ $(v, w \in W)$ if there exists a sequence $v = v_0, v_1, \ldots, v_n = w$ such that $v_i \overset{\cdot}{\leq} v_{i+1}$ for all $0 \leq i \leq n-1$.

Now we return to the S-submodule M_T of P_λ, where $T \subseteq W$. By 5.1.3, if $w \in W$, then for all $w_i \in \underline{R}$ and all $\mu \in (\chi)$,

5.1.11
$$e_\lambda A_{(w),F}\, e_\mu A_{(w_i),F} = \begin{cases} e_\lambda A_{(w_i)(w),F} & \text{if } \mu = w\lambda \text{ and } \ell(w_i w) > \ell(w) \\ 0 & \text{if either } \mu \neq w\lambda \text{ or } \ell(w_i w) < \ell(w) . \end{cases}$$

Definition A subset T of W is said to be "good" if $w \in T$ implies $v \in T$ for all $v \in W$ with $w \leq v$.

Lemma 5.1.12 For a given $T \subseteq W$, the S-submodule M_T of P_λ is an FS_χ-submodule if and only if T is "good".

Proof Suppose that $M_T = \sum_{w \in T} F.e_\lambda A_{(w)}$ is FS_χ-submodule of P_λ. Since FS_χ is generated as an F-algebra by $\{e_\mu A_{(w_i),F} \mid \mu \in (\chi), w_i \in \underline{R}\}$, it follows that if $w \in T$ then $e_\lambda A_{(w),F}\, e_\mu A_{(w_i),F} \in M_T$, for all $\mu \in (\chi)$ and all $w_i \in \underline{R}$. In particular, if $\mu = w\lambda$ and $\ell(w_i w) > \ell(w)$, then by 5.1.11

$$e_\lambda A_{(w),F}\, e_\mu A_{(w_i),F} = e_\lambda A_{(w_i)(w),F} \in M_T \quad .$$

But since $e_\lambda A_{(w_i)(w),F}$ is a scalar multiple of $e_\lambda A_{(w_i w),F}$, it follows that $e_\lambda A_{(w_i w),F} \in M_T$, for every $w_i \in \underline{R}$ with $\ell(w_i w) > \ell(w)$. Hence $w_i w \in T$ for all $w_i \in \underline{R}$ with $\ell(w_i w) > \ell(w)$, or equivalently T contains all the elements $v \in W$ with $w \dot{\leq} v$. Consequently T contains all the elements $v \in W$ with $w \leq v$, and hence T is "good". Conversely if T is "good" then $w \in T$ implies that $w_i w \in T$, for all $w_i \in \underline{R}$ with $\ell(w_i w) > \ell(w)$, and reversing the above argument, we have

$$e_\lambda A_{(w),F}\, e_\mu A_{(w_i),F} \in M_T \quad \text{for all} \quad w_i \in \underline{R} \, .$$

Hence M_T is an FS_χ-submodule of P_λ . □

CHAPTER 6. The p-adic Modules $Y(\chi,J)$

We saw in 4.2.2(i) that if $\chi \in \tilde{H}$ then

6.0.1 $$FY_\chi = e_{\bar\chi}(FY) = \sum^{\oplus}_{J\subseteq P(\chi)} FY(\chi,J) \ ,$$

where the FG-modules $FY(\chi,J)$ are pairwise non-isomorphic indecomposable. Let

6.0.2 $$e_{\bar\chi} = \sum_{J\subseteq P(\chi)} e(\chi,J)$$

be the orthogonal idempotent decomposition of $e_{\bar\chi}$ in $E(FY_\chi)$ which corresponds to the decomposition 6.0.1. It follows from 4.1.9. that

6.0.3 $$1_{E(FY)} = \sum_{\substack{\chi\in\tilde{H}\\ J\subseteq P(\chi)}} e(\chi,J)$$

is an orthogonal idempotent decomposition of $1_{E(FY)}$ in $E(FY)$. Since $E(Y)$ is complete with respect to the $\pi E(Y)$-adic topology, we can lift the equations 6.0.2 and 6.0.3 to get orthogonal idempotent decompositions

6.0.4 $$e_\chi = \sum_{J\subseteq P(\chi)} e^0(\chi,J) \ ,$$

6.0.5 $$1_{E(Y)} = \sum_{\substack{\chi\in\tilde{H}\\ J\subseteq P(\chi)}} e^0(\chi,J)$$

of e_χ and $1_{E(Y)}$, respectively, in $E(Y)$. The idempotent $e^0(\chi,J)$ in $E(Y)$ is such that

6.0.6 $$\overline{e^0(\chi,J)} = e(\chi,J)$$

for all admissible G-pairs (χ, J) .

Definition We write $e^{o}(\chi,J)(Y) = Y(\chi,J)$.

Since $e^{o}(\chi,J)$ is primitive idempotent in $E(Y)$, for all admissible G-pairs (χ,J) , $\underline{Y}(\chi,J) = e^{o}(\chi,J)(Y)$ is an indecomposable RG-lattice. It is clear that $Y(\chi,J) = FY(\chi,J)$, and that the $Y(\chi,J)$ are pairwise non-isomorphic RG-lattices since the $FY(\chi,J)$ are pairwise non-isomorphic.

It follows from 6.0.4 and 6.0.5 that

6.0.7 $$Y_\chi = e_\chi(Y) = \sum_{J\subseteq P(\chi)}^{\oplus} Y(\chi,J) \ , \quad \text{and}$$

6.0.8 $$Y = \sum_{\chi\in\tilde{H}}^{\oplus} e_\chi(Y) = \sum_{\substack{\chi\in\tilde{H}\\ J\subseteq P(\chi)}}^{\oplus} Y(\chi,J) \ .$$

In this chapter we use the results of N. Tinberg, introduced in §4.2, to calculate the characters of the RG-lattices $Y(\chi,J)$. Similar calculations were done independently by P. Sin [PS].

Replacing G by G_J , $J \subseteq \underline{R}$, in 6.0.3, we get (see §4.2)

$$1_{E(FY_J)} = \sum_{\substack{\chi\in\tilde{H}\\ S\subseteq P_J(\chi)}} e_J(\chi,S) \ .$$

Therefore we have a decomposition of the RG_J-lattice Y_J , similar to 6.0.8; namely

$$Y_J = \sum_{\substack{\chi\in\tilde{H}\\ S\subseteq P_J(\chi)}} Y_J(\chi,S) \ .$$

Since $Y_J \cong R_U^{G_J}$, where R_U is the trivial RU-lattice, and since

$(R_U^{G_J})^G = R_U^G = Y$, it follows from 6.0.8 (using Krull-Schmidt theorem) that $Y_J(\chi,S)^G$, as a component(*) of R_U^G, is isomorphic to sum of components, each of the form $Y(\chi,L)$, say

$$Y_J(\chi,S)^G \cong \sum_{L\subseteq P(\chi)} m_L\, Y(\chi,L) \; , \qquad (m_L \in \mathbb{Z}_{\geq 0}) \; .$$

Since the inducing functor $\mathrm{Ind}_{G_J}^G$ commutes with the reduction mod πR , we have

6.0.9 $$FY_J(\chi,S)^G \cong \sum_{L\subseteq P(\chi)} m_L\, FY(\chi,L) \; .$$

Comparing 6.0.9 with 4.2.3 we are able to calculate the m_L ; this gives

<u>Lemma 6.0.10</u> Let $J \subseteq \underline{R}$ and let (χ,S) be an admissible G_J-pair. Then,

$$Y_J(\chi,S)^G \cong \sum^{\oplus}_{\substack{L\subseteq P(\chi)\\ S=L\cap J}} Y(\chi,L) \; . \qquad \square$$

Putting $J = P(\chi)$ in 6.0.10 we have

6.0.11 $$Y(\chi,S) \cong Y_{P(\chi)}(\chi,S)^G \; ,$$

for every admissible G-pair (χ,S) . Therefore, in order to determine the character of the RG-lattice $Y(\chi,S)$, we only need to calculate the character of $Y_{P(\chi)}(\chi,S)$.

(*) The components of an RG-lattice M are the direct RG-summands of M .

Hence we assume that $\chi \in \hat{H}$ is such that $P(\chi) = \underline{R}$. Using 6.0.10 we have

6.0.12 $$Y_J(\chi,J)^G \cong \sum_{J \subseteq L \subseteq \underline{R}}^{\oplus} Y(\chi,L) \quad \text{for all} \quad J \subseteq \underline{R} \ .$$

Lemma 6.0.13 For any $J \subseteq \underline{R}$, $Y_J(\chi,J)$ is a one-dimensional RG_J-lattice.

Proof The lemma follows from the fact that $\overline{Y_J(\chi,J)} = FY_J(\chi,J)$ is one-dimensional FG_J-module ([NT2], lemma 4.10).

□

The one-dimensional FG_J-module $FY_J(\chi,J)$ $(J \subseteq \underline{R})$ is generated by an element $m_J(\chi,J) \in FY_J(\chi,J)$ (see [NT2], proof of 2.10), with the G_J-action on $m_J(\chi,J)$ given by

6.0.14
$$\begin{aligned} um_J(\chi,J) &= m_J(\chi,J) && \text{for all} \quad u \in U \ , \\ (w_i)m_J(\chi,J) &= m_J(\chi,J) && \text{for all} \quad w_i \in J \ , \quad \text{and} \\ hm_J(\chi,J) &= \chi(h)m_J(\chi,J) && \text{for all} \quad h \in H \ . \end{aligned}$$

Let $\rho_{\chi,J}$ be the one-dimensional character of G_J afforded by $FY_J(\chi,J)$. By 6.0.14 $\rho_{\chi,J}(g) \in (F^{\times})_c$ for all $g \in G_J$, where c is the exponent of H (see 4.1.4). Therefore $\rho_{\chi,J}$ can be lifted to a K-character $\rho^o_{\chi,J}$ of G_J by means of the map Γ of 4.1.4; i.e. $\rho^o_{\chi,J}(g) := \Gamma(\rho_{\chi,J}(g))$, for all $g \in G_J$. If θ is a k-character of G and $J \subseteq \underline{R}$, denote by $\theta|_{G_J}$ the restriction of θ to the parabolic subgroup G_J .

Lemma 6.0.15 (i) The RG_J-lattice $Y(\chi,\underline{R})$ affords the character $\rho^o_{\chi,\underline{R}}$.

(ii) $\rho^{o}_{\chi,\underline{R}}\big|_{G_J} = \rho^{o}_{\chi,J}$, for all $J \subseteq \underline{R}$.

<u>Proof</u> (i) Let T be the RG-lattice which affords the character $\rho^{o}_{\chi,\underline{R}}$. By 6.0.14, $\rho^{o}_{\chi,\underline{R}}\big|_{U} = 1_U$, and so $T\big|_U = R_U$, the trivial RU-lattice. Since U is a Sylow p-subgroup of G , T is U-projective ([CRI], Cor. 63.8). Hence T is a component of $(T\big|_U)^G = R_U^G \cong Y$. Therefore $T \cong Y(\chi',S)$ for some admissible G-pair (χ',S) . But then the FG-module $FT = F \otimes_R T$ should afford the character $\rho_{\chi,R}$ (since $\rho^{o}_{\chi,\underline{R}}$ is the lift of $\rho_{\chi,\underline{R}}$) . Hence $FT = FY(\chi',S) = FY(\chi,\underline{R})$, and so $(\chi',S) = (\chi,\underline{R})$ and $T = Y(\chi,\underline{R})$.

(ii) Clear from 6.0.14.

For $L \subseteq \underline{R}$, let η_L be the character afforded by the RG-lattice $Y(\chi,L)$. By 6.0.12, we have

6.0.16 $$\rho^{o\,G}_{\chi,J} = \sum_{J\subseteq L\subseteq \underline{R}} \eta_L \qquad \text{for all } J \subseteq \underline{R} .$$

Let $\lambda = (\rho^{o}_{\chi,\underline{R}})^{-1} : G \to K^{\times}$. Since $\rho^{o}_{\chi,\underline{R}}\big|_{G_J} = \rho^{o}_{\chi,J}$ (6.0.15(ii)), we have $\lambda\big|_{G_J}\rho^{o}_{\chi,J} = 1_{G_J}$, and so, by Frobenius reciprocity, we have

6.0.17 $$1_{G_J}^G = (\lambda\big|_{G_J}\,\rho^{o}_{\chi,J})^G = \lambda\,\rho^{o\,G}_{\chi,J} .$$

From the equations 6.0.16 and 6.0.17, we have

6.0.18 $$1_{G_J}^G = \sum_{J\subseteq L\subseteq \underline{R}} \eta_{1,L} \qquad \text{for all } J \subseteq \underline{R} ,$$

where $\eta_{1,L} = \lambda\eta_L$. Solving equation 6.0.18 for $\eta_{1,J}$ ([NB], Exercise 25, p.44-45), we get

6.0.19 $$\eta_{1,J} = \sum_{J \subseteq L \subseteq \underline{R}} (-1)^{|L \setminus J|} \, 1_{G_L}^{G} \quad .$$

Therefore we have

Proposition 6.0.20 Suppose $\chi \in \hat{H} = \mathrm{Hom}(H,K^{\times})$ is such that $P(\chi) = \underline{R}$ and let η_J , $J \subseteq \underline{R}$, be the character afforded by the RG-lattice $Y(\chi,J)$. Then

$$\eta_J = \rho^{o}_{\chi,\underline{R}} \left(\sum_{J \subseteq L \subseteq \underline{R}} (-1)^{|L \setminus J|} \, 1_{G_L}^{G} \right) .$$

Proof Clear from 6.0.19, since $\eta_{1,J} = \lambda \eta_J$ and $\lambda = (\rho^{o}_{\chi,\underline{R}})^{-1}$. □

In particular, if ϕ denotes the empty subset of $\underline{R}$ then

$$\eta_{\phi} = \rho^{o}_{\chi,\underline{R}} \left(\sum_{L \subseteq \underline{R}} (-1)^{|L|} \, 1_{G_L}^{G} \right)$$

$$= \rho^{o}_{\chi,\underline{R}} \cdot \mathrm{St}_G \qquad \text{(see 3.2.4),}$$

and

$$\eta_{\underline{R}} = \rho^{o}_{\chi,\underline{R}} \cdot 1_G = \rho^{o}_{\chi,\underline{R}} \quad .$$

CHAPTER 7. The Decomposition Numbers Of The System $(E(KY_\chi), E(Y_\chi), E(FY_\chi))$.

Let $\chi \in \hat{H}$ and let $J \subseteq \underline{R}$. Consider the R-algebra $E(Y_{\chi,J}) = \mathrm{End}_{RG_J}(Y_{\chi,J})$, Where $Y_{\chi,J} = e^J(Y_J)$, and $e^J_\chi = \frac{1}{|H|} \sum_{h\in H} \chi(h^{-1})A^J_h$. Since e^J_χ is an idempotent in $E(Y_J)$, $Y_{\chi,J}$ is a direct RG_J-summand of Y_J . Since Y_J is p-endostable RG_J-lattice, it follows (see [G2], Lemma 2.4) that $Y_{\chi,J}$ is also a p-endostable RG_J-lattice. Therefore if $k \in \{K,F\}$ then $k \otimes_R E(Y_{\chi,J}) \cong E(kY_{\chi,J})$, and so we may regard $E(Y_{\chi,J})$ as an R-order in $E(KY_{\chi,J})$. Let $E_{\chi,J} := e^J_\chi\, E(Y_J)e^J_\chi$, $KE_{\chi,J} := e^J_\chi\, E(KY_J)e^J_\chi$, and $FE_{\chi,J} = e^J_{\bar{\chi}}\, E(FY_J)e^J_{\bar{\chi}}$ (note that if $k \in \{K,F\}$ then $kE_{\chi,J} \cong E(kY_{\chi,J}))$. If $J = \underline{R}$, we write $Y_{\chi,\underline{R}} = Y_\chi$, $E_{\chi,\underline{R}} = E_\chi$ and if $k \in \{K,F\}$, we write $kY_{\chi,\underline{R}} = kY_\chi$ and $kE_{\chi,\underline{R}} = kE_\chi$. We saw in §4.2 that if $\chi \in \hat{H}$, then

$$FY_\chi = \sum_{S\subseteq P(\chi)}^{\oplus} FY(\chi,S) ,$$

where $FY(\chi,S)$ are mutually non-isomorphic indecomposable FG-modules. Consequently the simple modules for the F-algebra FE_χ (which is isomorphic to $E(FY_\chi))$ are all one-dimensional indexed by the set $\{(\chi,S) \mid S \subseteq P(\chi)\}$ (see [CRII], Prop. (6.3), p.120). We denote by $\mathcal{S}(\chi,S)$ the one-dimensional right FE_χ-module which corresponds to the pair (χ,S) .

In Chapter 7 we consider the decomposition numbers of the system $(KE_\chi, E_\chi, FE_\chi)$ using a recent theorem of Green ([G2], Thm. 4.2), which relates the decomposition numbers of the system $(KE_\chi, E_\chi, FE_\chi)$ to the multiplicities of the simple components of KY_χ. The next section will contain a parametrization of the simple KG-components of KY_χ.

§7.1 The irreducible characters of G

Let $G = (G,B,N,\underline{R},U)$ be a finite group with a split BN-pair of characteristic p.

The Levi decomposition ([RC], §2.6) For each $J \subseteq \underline{R}$, let w_J denote the unique element of W_J of maximal length. Let $U_J = U \cap U^{(w_J)}$ and $L_J = \langle H, X_\alpha ; \alpha \in \Phi_J \rangle$. U_J is a normal p-subgroup of the standard parabolic subgroup G_J and G_J has a decomposition $G_J = L_J U_J$ as a semidirect product of L_J and U_J. Moreover $B_J = HU^-_{w_J}$ and N_J form a split BN-pair of L_J of rank $|J|$, where N_J is the inverse image of W_J under t. The above decomposition of G_J is called the Levi decomposition and L_J is called Levi subgroup of G_J.

Let $\mathrm{ch}_K G_J$ denote the set of all characters of G_J over K. Since $L_J \cong G_J/U_J$, each character θ of L_J, over K, gives a character $\theta_{G_J} \in \mathrm{ch}_K G_J$ having U_J in its kernel, in fact θ_{G_J} is given by

$$\theta_{G_J}(\ell u) := \theta(\ell), \quad \text{for all } u \in U_J, \ \ell \in L_J .$$

For $\theta \in \mathrm{ch}_K G_J$, $J \subseteq R$, let $\theta^G_{G_J}$ be the induced character $\mathrm{Ind}^G_{G_J} \theta_{G_J}$ of θ_{G_J}.

Definitions (See [RC], §9.1)

(i) $\chi \in ch_K G$ is called cuspidal if for all $J \subsetneq \underline{R}$ and all $\theta \in ch_K L_J$ $\langle \chi, \theta_{G_J}^G \rangle_G = 0$, i.e. χ does not appear as a component of $\theta_{G_J}^G$.

(ii) If J_1 and J_2 are subsets of $\underline{R}$, we say that J_1 and J_2 are associated if $w(\pi_{J_1}) = \pi_{J_2}$ for some $w \in W$.

(iii) For each $J \subseteq \underline{R}$ let $\Omega_J = \{w \in W \mid w(\pi_J) = \pi_J\}$.

It follows (see for example [RC], Prop. 9.2.2) that if $w \in \Omega_J$ then ${}^{(w)}L_J = L_J$. Therefore the group Ω_J acts on the set $ch_K L_J$ as follows: if $\chi \in ch_K L_J$ and $w \in \Omega_J$ then $w\chi \in ch_K L_J$ is given by

$$w\chi(\ell) := \chi(\ell^{(w)}) \quad \text{for all} \quad \ell \in L_J .$$

The Harish-Chandra theory for dividing the irreducible characters of G into classes suggests that one can find every irreducible character θ of G as a component of $\chi_{G_J}^G$, for some $J \subseteq \underline{R}$ and some cuspidal character $\chi \in ch_K(L_J)$.

The following summarizes Harish-Chandra theory.

Theorem 7.1.1 ([RC], Theorem 9.2.3)

(i) Each irreducible character θ of G appears as a component of $\chi_{G_J}^G$ for some cuspidal character χ of some Levi subgroup L_J of G .

(ii) Take one $J \subseteq \underline{R}$ for each class of associated subsets of $\underline{R}$ and one cuspidal character χ of L_J in each Ω_J-orbit of cuspidal characters. If we take all irreducible components of $\chi_{G_J}^G$, then we get each irreducible character of G just once. $\square$

In [HL], Howlett and Lehrer have given a method of parametrizing the irreducible components of $\chi_{G_J}^G$, where χ is an irreducible character of L_J, by studying the endomorphism algebra of a KG-module affording $\chi_{G_J}^G$. The K-basis elements of this endomorphism algebra are indexed by the subgroup $W_{\chi,J} := \{w \in \Omega_J \mid w\chi = \chi\}$ of W. Using some specializations of a certain generic Hecke algebra, Howlett and Lehrer showed that the irreducible characters of G which appear in $\chi_{G_J}^G$ are in 1-1 correspondence with the irreducible characters of the group algebra $(KW_{\chi,J})_\mu$ twisted by some cocycle μ (see [HL], Corollary 5.5).

However, for our purpose we will take $J = \phi$, the empty subset of $\underline{R}$, hence $L_J = L_\phi = H$, and $G_\phi = B$, the standard Borel subgroup of G. It is clear that all irreducible characters of H are cuspidal. If $\chi \in \hat{H}$, then χ_B^G is the character of G afforded by the KG-module KY_χ $(= e_\chi(KY))$. Recall that the K-algebra $KE_\chi = e_\chi E(KY)e_\chi$ is isomorphic to $E(KY_\chi)$ and has K-basis $\{e_\chi A_{(w),K}, w \in W_\chi\}$, where $W_\chi = W_{\chi,\phi} = \{w \in W \mid w\chi = \chi\}$ (note that $\Omega_\phi = W$).

Remark 7.1.2 The group $W_{\chi,J}$ is not, in general, a reflection subgroup of W. However, the theory of Howlett and Lehrer [LH] shows that $W_{\chi,J}$ has a decomposition $W_{\chi,J} = M_{\chi,J}M'_{\chi,J}$ as a semidirect product, where $M_{\chi,J}$ is a reflection normal subgroup of $W_{\chi,J}$.

§7.2 The Decomposition Numbers As Multiplicities Of Ordinary Characters

For $\chi \in \tilde{H}$, let I be an index set for the complete set of irreducible characters of the twisted group algebra $(KW_\chi)_\mu$. Then, by Howlett-Lehrer theory, I is also an index set for the set $\{\theta_i\ ,\ i \in I\}$ of irreducible characters of G which appear in χ_B^G.

For each $i \in I$, let X_i be an RG-lattice such that KX_i is a KG-module which affords the character θ_i. Then KX_i is a simple component of KY_χ, for all $i \in I$. The K-space $(KY_\chi, KX_i)_{KG}$, $i \in I$, has a natural structure of a right $E(KY_\chi)$-module, hence a right KE_χ-module (since $KE_\chi \cong E(KY_\chi)$). In fact $(KY_\chi, KX_i)_{KG}$ is a simple KE_χ-module (see [CRII], Prop. 6.3, p.120), for all $i \in I$, and the set

$$\{(KY_\chi, KX_i)_{KG}\ ,\ i \in I\}$$

is a complete set of simple right KE_χ-modules. For each $i \in I$, X_i is an R-form of the KG-module KX_i.

Consider the R-lattice $(Y_\chi, X_i)_{RG}$, $i \in I$. $(Y_\chi, X_i)_{RG}$ is a right $E(Y_\chi)$-lattice, hence it is a right E_χ-lattice. We identify X_i with an R-submodule of KX_i by identifying $x \in X_i$ with $1_K \otimes x \in KX_i$. Similarly Y_χ can be identified with an R-submodule of KY_χ. Consequently the R-lattice $(Y_\chi, X_i)_{RG}$ can be regarded as a subset of $(KY_\chi, KX_i)_{KG}$ by identifying $f \in (Y_\chi, X_i)_{RG}$ with the unique K-map $f' \in (KY_\chi, KX_i)_{KG}$ which coincide with f on Y_χ. Since $K \otimes_R (Y_\chi, X_i)_{RG} \cong (KY_\chi, KX_i)_{KG}$, we may then regard the right E_χ-lattice $(Y_\chi, X_i)_{RG}$ as an R-form of the right KE_χ-module $(KY_\chi, KX_i)_{KG}$.

If $f \in (Y_\chi, X_i)_{RG}$, let $\bar{f} = 1_F \underset{R}{\otimes} f$, and let

$$\overline{(Y_\chi, X_i)}_{RG} = F \underset{R}{\otimes} (Y_\chi, X_i)_{RG} = \{\bar{f}\ ,\ f \in (Y_\chi, X_i)_{RG}\} \ .$$

(Note that in general $\overline{(Y_\chi, X_i)}_{RG} \neq (FY_\chi, FX_i)_{FG}$, since we are not assuming that the pair Y_χ, X_i is p-stable.) It is clear that if $f \in (Y_\chi, X_i)$ and $g \in E(Y_\chi)$, then $\overline{fg} = \bar{f}\,\bar{g}$, therefore $\overline{(Y_\chi, X_i)}_{RG}$ is a right $E(FY_\chi)$-module (since $F \underset{R}{\otimes} E(Y_\chi) \cong E(FY_\chi)$). Hence $\overline{(Y_\chi, X_i)}_{RG}$ is a right FE_χ-module, which is a p-modular version of the simple right KE_χ-module $(KY_\chi, KX_i)_{KG}$.

<u>Definition</u> (R. Brauer, see [CRII], p.413) For every $S \subseteq P(\chi)$, and all $i \in I$, let $d_{i,(\chi,S)}$ denote the multiplicity with which the simple right FE_χ-module $\mathcal{S}(\chi,S)$ appears as a composition factor of $\overline{(Y_\chi, X_i)}$. The numbers $d_{i,(\chi,S)}$ are called <u>the p-decomposition numbers</u> of the system $(KE_\chi, E_\chi, FE_\chi)$.

Let $S \subseteq P(\chi)$ and let $P(\chi,S)$ be a projective cover of $\mathcal{S}(\chi,S)$. We may take $P(\chi,S) = e(\chi,S)FE_\chi$, where $e(\chi,S)$ is the primitive idempotent given in 6.0.2. Then, for all $i \in I$, we have (see [CRII], Exercise 10(a), p.71)

7.2.2 $$d_{i,(\chi,S)} = \dim_F (P(\chi,S), \overline{(Y_\chi, X_i)})_{FG}$$

$$= \dim_F \overline{(Y_\chi, X_i)}\, e(\chi,S) \ .$$

On the other hand, let $e^o(\chi,S)$ be the lift of $e(\chi,S)$ in E_χ. $e^o(\chi,S)$ can be regarded as an idempotent in KE_χ.

Since $e^0(\chi,S)\,(Y_\chi) = Y(\chi,S)$ by definition, we have $e^0(\chi,S)(KY_\chi) = KY(\chi,S)$, and since $KY(\chi,S)$ is a KG-submodule of KY_χ , we then have

$$KY(\chi,S) = \sum_{i\in I}^{\oplus} d^*_{i,(\chi,S)}\ KX_i\ ,\quad d^*_{i,(\chi,S)} \in \mathbb{Z}_{\geq 0}\ .$$

A more general setting of the following theorem was given by Green [G2].

<u>Theorem 7.2.3</u> (Green [G2] Theorem 4.2) If $\chi \in \tilde{H}$, then

$$d_{i,(\chi,S)} = d^*_{i,(\chi,S)}$$

for all $i \in I$ and all $S \subseteq P(\chi)$. □

<u>Remark</u> The above theorem provides a formula for the multiplicities of the simple components of KY_χ , $\chi \in \tilde{H}$, provided one knows the decomposition matrix of the system $(KE_\chi, E_\chi, FE_\chi)$. In fact if

$$D_\chi := \begin{array}{c} \\ i \end{array}\!\!\begin{array}{c} (\chi,S) \\ \left(\begin{array}{c} \vdots \\ \cdots\ \cdots\ d_{i,(\chi,S)} \end{array}\right) \end{array} \quad ;\quad i \in I \text{ and } S \subseteq P(\chi)$$

is the decomposition matrix of the system $(KE_\chi, E_\chi, FE_\chi)$, then by 7.2.3,

$$[KX_i \mid KY_\chi] = [KX_i \mid \sum_{S\subseteq P(\chi)}^{\oplus} KY(\chi,S)]$$

$$= \sum_{S\subseteq P(\chi)} d_{i,(\chi,S)}$$

for all $i \in I$, where $[KX_i \mid KY_\chi]$ denotes the multiplicity of KX_i as a component of KY_χ.

* * * * *

Now let $J \subseteq \underline{R}$ be such that $W_\chi \leq W_J$. Let $\xi = \chi_B^{G_J}$ and let $\langle\ ,\ \rangle_G$ denote the usual scalar product on $\mathrm{ch}_K G$ (see for example [CRII], p.210).

<u>Lemma 7.2.4</u> $\langle \xi^G, \xi^G \rangle_G = \langle \xi, \xi \rangle_{G_J}$.

<u>Proof</u> By Frobenius reciprocity formula and Mackey decomposition (see [CRII], p.237), we have

$$\begin{aligned}\langle \xi^G, \xi^G \rangle_G &= \langle (\chi_B^{G_J})^G , (\chi_B^{G_J})^G \rangle_G \\ &= \langle \chi_B^G , \chi_B^G \rangle \\ &= \sum_{w \in W} \langle w\chi, \chi \rangle_{{}^wB \cap B} .\end{aligned}$$

Since $H \leq {}^wB \cap B$, for all $w \in W$, $\langle w\chi, \chi \rangle_{{}^wB \cap B} = 0$ unless $w\chi = \chi$; i.e. unless $w \in W_\chi$. But since $W_\chi \leq W_J$, we have

$$\begin{aligned}\langle \xi^G, \xi^G \rangle_G &= \sum_{w \in W_\chi} \langle w\chi, \chi \rangle_{{}^wB \cap B} \\ &= \sum_{w \in W_J} \langle w\chi\ \chi \rangle_{{}^wB \cap B} \\ &= \langle \chi_B^{G_J}, \chi_B^{G_J} \rangle_{G_J} = \langle \xi, \xi \rangle_{G_J} .\end{aligned}$$

□

Suppose that $\xi = \sum_i m_i \xi_i$, where $m_i \in \mathbb{Z}_0$, and ξ_i is an irreducible character of G_J for all i , with $\xi_i = \xi_j$ only if $i = j$, then we have

<u>Corollary 7.2.5</u> ξ_i^G is irreducible character of G for all i and $\xi_i^G = \xi_j^G$ only if $i = j$.

<u>Proof</u>
$$<\xi^G,\xi^G>_G = \sum_{i,j} m_i m_j <\xi_i^G,\xi_j^G>_G$$
$$= \sum_i m_i^2 <\xi_i^G,\xi_i^G>_G + \sum_{i \neq j} m_i m_j <\xi_i^G,\xi_j^G>_G .$$

But by 7.2.4

$$<\xi^G,\xi^G>_G = <\xi,\xi>_{G_J} = \sum_i m_i^2 .$$

Therefore we must have,

$$<\xi_i^G,\xi_i^G> = 1 \quad \text{for all } i , \quad \text{and} \quad <\xi_i^G,\xi_j^G>_G = 0$$

if $i \neq j$. It follows from the orthogonality relations (see [CRII], 9.23) that ξ_i^G is irreducible for all i , and that $\xi_i^G \neq \xi_j^G$ if $i \neq j$.

□

It follows from 7.2.4 and 7.2.5 that the map $\xi_i \to \xi_i^G$ gives a 1-1 correspondence between the set of irreducible characters of G_J which appear in $\chi_B^{G_J}$ and the set of irreducible characters of G which appear in χ_B^G . Moreover, the corresponding characters under this correspondence appear with the same multiplicity. So we may assume that, for every $i \in I$,

$\xi_i^G = \theta_i$ and hence we have

$$\chi_B^{G_J} = \sum_{i \in I} m_i \xi_i \quad , \quad m_i \in \mathbb{Z}_{>0} \ , \ \text{and}$$

$$\chi_B^G = \xi^G = \sum_{i \in I} m_i \xi_i^G = \sum_{i \in I} m_i \theta_i \ .$$

Recall that the KG_J-module $KY_{\chi,J} = e_\chi^J(Y_J) = KG_J[U]\beta_\chi$ affords the character $\chi_B^{G_J}$. For every $i \in I$, let $X_{i,J}$ be an RG_J-lattice such that $KX_{i,J}$ is a simple KG_J-module which affords the character ξ_i. Since $\xi_i^G = \theta_i$ $(i \in I)$ and since θ_i is afforded by the KG-module KX_i, it follows that $KX_{i,J}^G \cong KX_i$, for all $i \in I$.

We would like to compare the decomposition numbers of the two systems $(KE_\chi, E_\chi, FE_\chi)$ and $(KE_{\chi,J}, E_{\chi,J}, FE_{\chi,J})$, where $J \subseteq \underline{R}$ with $W_\chi \leq W_J$. Let D_χ, $D_{\chi,J}$ be the decomposition numbers of the systems $(KE_\chi, E_\chi, FE_\chi)$, $(KE_{\chi,J}, E_{\chi,J}, FE_{\chi,J})$, respectively. Since the number of irreducible characters of G which appear as components of χ_B^G is equal to the number of irreducible characters of G_J which appear as components of $\chi_B^{G_J}$, D_χ and $D_{\chi,J}$ have the same number of rows, namely $|I|$. Moreover, since $W_{P(\chi)} \leq W_\chi$ (see 4.2.1), it follows that $P(\chi) \subseteq J$, and so the simple FE_χ- and $FE_{\chi,J}$-modules are indexed by all subsets of $P(\chi)$. Therefore D_χ and $D_{\chi,J}$ have the same number of columns, namely $2^{|P(\chi)|}$. Hence the decomposition matrices D_χ and $D_{\chi,J}$ have the same size.

Recall that the R-order $E_{\chi,J}$ has R-basis $\{e_\chi^J A_{(w)}^J \ , \ w \in W_\chi\}$. The map $e_\chi^J A_{(w)}^J \to e_\chi A_{(w)}$, $w \in W_\chi$, defines an injective R-algebra map

$a : E_{\chi,J} \to E_\chi$ (see [NT2], p.511). But since $\mathrm{rank}_R\, E_{\chi,J} = \mathrm{rank}_R\, E_\chi = |W_\chi|$, it follows that a is an R-isomorphism. Hence we have:

<u>Lemma 7.2.6</u> If $\chi \in \tilde{H}$, and $J \subseteq \underline{R}$ is such that $W_\chi \leq W_J$, then the R-orders $E_{\chi,J}$ and E_χ are isomorphic. □

Since $W_\chi \leq W_J$ and $W_{P(\chi)} \leq W_\chi$ (see 4.2.1), it follows that $P(\chi) \subseteq J$. Hence, from 6.0.7, we have

7.2.7 $$Y_\chi = \sum^{\oplus}_{S\subseteq P(\chi)} Y(\chi,S) , \quad \text{and}$$

7.2.8 $$Y_{\chi,J} = \sum^{\oplus}_{S\subseteq P(\chi)} Y_J(\chi,S) .$$

By 6.0.10, we also have $Y_J(\chi,S)^G \cong Y(\chi,S)$, for all $S \subseteq P(\chi)$.

Let

7.2.9 $$e_\chi = \sum_{S\subseteq P(\chi)} e^o(\chi,S) , \quad \text{and}$$

7.2.10 $$e^J_\chi = \sum_{S\subseteq P(\chi)} e^o_J(\chi,S)$$

be the orthogonal primitive idempotent decomposition of e_χ and e^J_χ in E_χ and $E_{\chi,J}$, respectively, which correspond to the decompositions 7.2.7 and 7.2.8. Since $a(e^J_\chi) = e_\chi$, we may arrange the decompositions 7.2.9 and 7.2.10 so that $a(e^o_J(\chi,S)) = e^o(\chi,S)$ for all $S \subseteq P(\chi)$. Since $e^o_J(\chi,S)$ is an R-combination of the elements $\{e^J_\chi A^J_{(w)} , w \in W_\chi\}$, and since $e^J_\chi A^J_{(w)}([U]\beta_\chi) = \beta_\chi[U(w)U] = e_\chi A_{(w)}([U]\beta_\chi)$, for all $w \in W_\chi$, it follows that

$$e_J^O(\chi,S)\ ([U]\beta_\chi) = e^O(\chi,S)\ ([U]\beta_\chi) \quad \text{for all} \quad S \subseteq P(\chi)\ .$$

Therefore for every $S \subseteq P(\chi)$, we have

$$\begin{aligned} KY(\chi,S) &= e^O(\chi,S)(KY_\chi) \\ &= e^O(\chi,S)(KG[U]\beta_\chi) \\ &= KG\ e^O(\chi,S)([U]\beta_\chi) \\ &= KG\ e_J^O(\chi,S)([U]\beta_\chi) \\ &\cong KY_J(\chi,S)^G \qquad \text{(see [CRII], Prop. 11.21).} \end{aligned}$$

For $i \in I$, let $d^J_{i,(\chi,S)^*}$ denote the multiplicity of $KX_{i,J}$ as a composition factor of $KY_J(\chi,S)$.

<u>Lemma 7.2.11</u> Let χ and J be as in 7.2.6. Then

$$d^J_{i,(\chi,S)^*} = d_{i,(\chi,S)^*} \quad \text{for all} \quad S \subseteq P(\chi)\ .$$

<u>Proof</u> $$\begin{aligned} d_{i,(\chi,S)^*} &= [KX_i \mid KY(\chi,S)] \\ &= [KX^G_{i,J} \mid KY_J(\chi,S)^G] \\ &= [KX_{i,J} \mid KY_J(\chi,S)] \\ &= d^J_{i,(\chi,S)^*}\ . \end{aligned}$$ □

Since D_χ and $D_{\chi,J}$ are of the same size, it follows from 7.2.3 and 7.2.11 that, under a suitable arrangement, the matrices:

$$D_\chi = \begin{array}{cc} & S \\ & \vdots \\ i & \left(\begin{array}{c} \\ \ldots d_{i,(\chi,S)^*} \\ \\ \end{array}\right) \end{array} \quad \text{and} \quad D_{\chi,J} = \begin{array}{cc} & S \\ & \vdots \\ i & \left(\begin{array}{c} \\ \ldots d^J_{i,(\chi,S)^*} \\ \\ \end{array}\right) \end{array}$$

are identical. Therefore we may replace the group G in theorem 7.2.3 by G_J where J is any subset of $\underline{R}$ with $W_\chi \leq W_J$.

Now let J be any subset of $\underline{R}$. Recall that $L_J = \langle B_J, N_J \rangle$, where $B_J = U^-_{w_J} H$, and $U^-_{w_J} \trianglelefteq B_J$. Therefore any $\chi \in \hat{H}$ can be extended to a character χ_{B_J} of B_J .

For $\chi \in \hat{H}$, let $Y^J_\chi := RL_J[U^-_{w_J}]\beta_\chi$, $KY^J_\chi := KL_J[U^-_{w_J}]\beta_\chi$ and $FY^J_\chi := FL_J[U^-_{w_J}]\beta_{\bar{\chi}}$. If $X \in \text{mod}^o RL_J$, let X_{G_J} denote X regarded as an RG_J-lattice by letting U_J acting trivially on X .

<u>Proposition 7.2.12</u> For all $J \subseteq \underline{R}$ and all $\chi \in \hat{H}$ we have:

$$(Y^J_\chi)_{G_J} \cong Y_{\chi,J} \quad \text{as } RG_J \text{ - lattices.}$$

<u>Proof</u> We have $G_J = L_J U_J$ and $L_J \leq B \leq G_J$. Hence $G_J = L_J B$, and so any left coset of B in G_J has the form ℓB for some $\ell \in L_J$. If $\ell_1, \ell_2 \in L_J$ then $\ell_1 B = \ell_2 B \iff \ell_2^{-1}\ell_1 \in B \cap L_J = B_J \iff \ell_1 B_J = \ell_2 B_J$. Therefore there is a bijection between the left B-cosets of G_J and the left B_J-cosets of L_J . In particular $\text{rank}_R Y_{\chi,J} = |G_J : B| = |L_J : B_J| = \text{rank}_R Y^J_\chi$. Let $\zeta : Y_{\chi,J} \to Y^J_\chi$ be the bijective R-map which sends $\ell[U]\beta_\chi$ to $\ell[U^-_{w_J}]\beta_\chi$ $(\ell \in L_J)$. If $u \in U_J$, and $\ell \in L_J$ then

$$\begin{aligned}
u\ell[U]\beta_\chi &= \ell\ell^{-1}u\ell[U]\beta_\chi \\
&= \ell\ell^{-1}u\ell\beta_\chi[U] \\
&= \chi(\ell^{-1}u\ell)\ \ell[U]\beta_\chi && \text{since } U_J \trianglelefteq G_J \\
&= \ell[U]\beta_\chi && \text{since } \chi|_{U_J} = 1_{U_J} .
\end{aligned}$$

Therefore U_J acts trivially on $Y_{\chi,J}$. Consequently the map $\zeta : Y_{\chi,J} \to (Y_\chi^J)_{G_J}$ is an RG_J-map, for if $g = \ell'u \in G_J$ $(\ell' \in L_J, u \in U_J)$, then

$$\begin{aligned}
\zeta(g\ell[U]\beta_\chi) &= \zeta(\ell'u\ell[U]\beta_\chi) \\
&= \zeta(\ell'\ell[U]\beta_\chi) \\
&= \ell'\ell[U^-_{w_J}]\beta_\chi \\
&= \ell'\zeta(\ell[U]\beta_\chi) \\
&= \ell'u\zeta(\ell[U]\beta_\chi) \\
&= g\zeta(\ell[U]\beta_\chi)
\end{aligned}$$

for all $\ell \in L_J$. Hence $Y_{\chi,J} \cong (Y^J)_{G_J}$ as RG_J-lattices. □

The R-lattices $E(Y_J)$ and $E(Y^J)$ (where $Y_J = RG_J[U]$ and $Y^J = RL_J[U^-_{w_J}]$, $J \subseteq \underline{R}$) have R-bases indexed by the elements of N_J . In fact $E(Y_J) = \sum^{\oplus}_{n\in N_J} R.A_n^J$ and $E(Y^J) = \sum^{\oplus}_{n\in N_J} R.A'^J_n$, where A_n^J and A'^J_n $(n \in N_J)$ are given by

$$A_n^J([U]) := [UnU] \quad \text{and} \quad A'_n([U^-_{w_J}]) := [U^-_{w_J}nU^-_{w_J}] \ .$$

Let $\chi \in \hat{H}$ and let $n \in N_J$ be such that $t(n) = w \in W_\chi$. Then by 4.1.14, the restriction of A_n^J to $Y_{\chi,J}$ induces an element of $E(Y_{\chi,J})$. Similarly, the restriction of A'^J_n to Y_χ^J induces an element of $E(Y_\chi^J)$. It is clear that the endomorphism algebra

$E(Y_\chi^J)$ (= $\mathrm{End}_{RL_J}(Y_\chi^J)$) is identical with $E((Y_\chi^J)_{G_J})$ (= $\mathrm{End}_{RG_J}((Y_\chi^J)_{G_J})$) .
Consider the diagram

$$\text{7.2.13}\qquad \begin{array}{ccc} Y_{\chi,J} & \xrightarrow{\ \zeta\ } & (Y_\chi^J)_{G_J} \\ {\scriptstyle A_n^J}\big\downarrow & & \big\downarrow{\scriptstyle A_n'^J} \\ Y_{\chi,J} & \xrightarrow[\ \zeta\ ]{} & (Y_\chi^J)_{G_J} \end{array} \quad ,$$

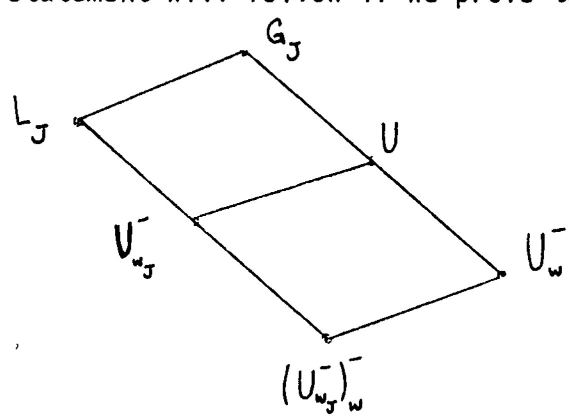
where ζ is the RG_J-map defined in the proof of 7.2.12.

<u>Lemma 7.2.14</u> The diagram of 7.2.13 is commutative.

<u>Proof</u> Since the maps of the diagram 7.2.13 are all RG_J-maps, it is enough to show that $\zeta A_n^J([U]\beta_\chi) = A_n'^J\zeta([U]\beta_\chi)$ or equivalently

$$\zeta([UnU]\beta_\chi)\ (= \zeta([U_w^-]n[U]\beta_\chi) = [U_{w_J}^- n U_{w_J}^-]\beta_\chi \ .$$

The last statement will follow if we prove that $U_w^- \cap U_{w_J}^- = (U_{w_J}^-)_w^-$.



We have $U_w^- \cap U_{w_J}^- = U \cap (U^-)^w \cap (U^-)^{w_J}$, and $(U_{w_J}^-)_w^- = U_{w_J}^- \cap (U_{w_J}^-)^{w_J w} =$
$= U \cap (U^-)^{w_J} \cap U^{w_J w} \cap (U^-)^w$. It is clear that $(U_{w_J}^-)_w^- \leq U_w^- \cap U_{w_J}^-$.
From 1.2.8(ii), we have

$$U_w^- \cap U_{w_J}^- = \langle X_\alpha \mid \alpha \in \Phi_J^+ , w(\alpha) < 0 \rangle$$

(note that $w_J(\alpha) < 0$, for all $\alpha \in \Phi_J^+$) .

If $X_\alpha \le U_w^- \cap U_{w_J}^-$ $(\alpha \in \Phi_J)$ then $X_\alpha \le U_{w_J}^-$ and $w(\alpha) < 0$ and hence, by 1.2.8(ii) again, $X_\alpha \le (U_{w_J}^-)_w^-$. Hence $(U_{w_J}^-)_w^- = U_w^- \cap U_{w_J}^-$.
This completes the proof of lemma 7.2.14.

□

The R-order $E(Y_{\chi,J})$ is isomorphic to $E_{\chi,J} = e_\chi^J E(Y_J)e_\chi^J$ which has R-basis $\{e_\chi^J A_{(w)}^J , w \in W_\chi \cap W_J\}$ (see the proof of 5.0.7).
Similarly $E(Y_\chi^J)$ is isomorphic to $e_\chi'^J E(Y^J)e_\chi'^J$ (where $e_\chi'^J = |H|^{-1} \Sigma \chi(h^{-1})A_h'^J$) which has R-basis $\{e_\chi'^J A_{(w)}'^J , w \in W_\chi \cap W_J\}$.

<u>Proposition 7.2.15</u> The map $\sigma : e_\chi^J A_{(w)}^J \mapsto e_\chi'^J A_{(w)}'^J$ $(w \in W_\chi \cap W_J)$ defines an R-algebra isomorphism between $e_\chi^J E(Y_J)e_\chi^J$ and $e_\chi'^J E(Y^J)e_\chi'^J$.
Consequently $E(Y_{\chi,J}) \cong E(Y_\chi^J)$ as R-algebras.

<u>Proof</u> σ is clearly an R-isomorphism. From lemma 7.2.14, we have a commutative diagram

$$\begin{tikzcd}
Y_{\chi,J} \arrow[r, "\zeta"] \arrow[d, "e_\chi^J A_{(w)}^J"'] & Y_\chi^J \arrow[d, "e_\chi'^J A_{(w)}'^J"] \\
Y_{\chi,J} \arrow[r, "\zeta"'] & Y_\chi^J
\end{tikzcd} ,$$

for all $w \in W_\chi \cap W_J$. Therefore the corresponding elements in the sets $\{e_\chi^J A_{(w)}^J , w \in W_\chi \cap W_J\}$ and $\{e_\chi'^J A_{(w)}'^J , w \in W_\chi \cap W_J\}$ under σ satisfy the same multiplication relations. Hence σ is an R-algebra isomorphism.

□

Let $J \subseteq \underline{R}$ and consider the KG_J-module $KY_{\chi,J}$. By 7.2.12, we have $KY_{\chi,J} \cong (KY_\chi^J)_{G_J}$. If X is a simple KG_J-component of $KY_{\chi,J}$ then U_J acts trivially on X, and X, regarded as KL_J-module, remains simple. Conversely if T is a simple KL_J-component of KY_χ^J then T_{G_J} is also simple. Therefore the simple KG_J-components of $KY_{\chi,J}$ are in 1-1 correspondence with the simple KL_J-components of KY_χ^J. Moreover, if $\{KX_{i,J} ; i \in I\}$ is a full set of simple KG_J-components of $KY_{\chi,J}$ and $\{KX_i^J ; i \in I\}$ is a full set of simple KL_J-components of KY_χ^J, then, under a suitable arrangement, we will have

7.2.16 $$KX_{i,J} \cong (KX_i^J)_{G_J} ,$$

for all $i \in I$. On the other hand if $W_\chi \leq W_J$, by 4.2.1, $P(\chi) \subseteq J$ and so the indecomposable FG_J, FL_J-summands of $FY_{\chi,J}$, FY_χ^J, respectively, are indexed by all subsets of $P(\chi)$. Let

$$FY_{\chi,J} = \sum_{S\subseteq P(\chi)}^{\oplus} FY_J(\chi,S) \qquad \text{and}$$

$$FY_\chi^J = \sum_{L\subseteq P(\chi)}^{\oplus} FY^J(\chi,L)$$

be the decomposition of $FY_{\chi,J}$, FY_χ^J into indecomposable FG_J, FL_J-modules, respectively.

For each $S \subseteq P(\chi)$, let $\psi_J(\chi,S)$, $\psi^J(\chi,S)$ be the one-dimensional F-character of $E(FY_{\chi,J})$, $E(FY_\chi^J)$, which correspond to $FY_J(\chi,S)$, $FY^J(\chi,S)$, respectively. For each $S \subseteq P(\chi)$, we have the following commutative diagram:

7.2.17
$$\begin{tikzcd}
E(Y_{\chi,J}) \arrow[rr, "\sigma"] \arrow[d] & & E(Y^J_\chi) \arrow[d] \\
E(FY_{\chi,J}) \arrow[rr, "Id_F \otimes \sigma"] \arrow[dr, "\psi_J(\chi,S)"'] & & E(FY^J_\chi) \arrow[dl, "\psi^J(\chi,S)"] \\
& F &
\end{tikzcd}$$

where σ is the R-algebra isomorphism defined in 7.2.15. Therefore, if $e_J(\chi,S)$ is a primitive idempotent of $E(FY_{\chi,J})$ corresponding to $\psi_J(\chi,S)$, then $Id_F \underset{R}{\otimes} \sigma\ (e_J(\chi,S))$ is a primitive idempotent of $E(FY^J_\chi)$ corresponding to $\psi^J(\chi,S)$. Hence, the map $Id_F \otimes \zeta : FY_{\chi,J} \to FY^J_\chi$ maps $FY_J(\chi,S) := e_J(\chi,S)(FY_{\chi,J})$ onto $(Id_F \underset{R}{\otimes} \sigma)(e_J(\chi,S))(FY^J_\chi) = FY^J(\chi,S)$, and so

$$FY_J(\chi,S) \cong (FY^J(\chi,S))_{G_J} \quad .$$

Let $e^o_J(\chi,S)$ be the lift in $E(Y_{\chi,J})$ of the primitive idempotent $e_J(\chi,S)$ and define $Y_J(\chi,S) := e^o_J(\chi,S)(Y_{\chi,J})$. $e^o_J(\chi,S)$ is a primitive and so $Y_J(\chi,S)$ is an indecomposable RG_J-lattice. Let ${}^Je^o(\chi,S) := \sigma(e^o_J(\chi,S))$ and define $Y^J(\chi,S) := {}^Je^o(\chi,S)(Y^J_\chi)$. It follows that ζ maps $Y_J(\chi,S)$ onto $Y^J(\chi,S)$ and so we have

7.2.18 $$Y_J(\chi,S) \cong (Y^J(\chi,S))_{G_J} \quad .$$

It follows that, if $KX_{i,J}$ $(i \in I)$ is a simple KG_J-component of $KY_{\chi,J}$, then for every $S \subseteq P(\chi)$, we have

7.2.19 $$\begin{aligned} d^{J\,*}_{i,(\chi,S)} &= [KX_{i,J} \mid KY_J(\chi,S)] \\ &= [(KX^J_i)_{G_J} \mid (KY^J(\chi,S))_{G_J}] \quad \text{by 7.2.16 and 7.2.18} \\ &= [KX^J_i \mid KY^J(\chi,S)] \quad . \end{aligned}$$

Let D_χ^J be the decomposition matrix of the system $(E(KY_\chi^J), E(Y_\chi^J), E(FY_\chi^J))$. It follows from 7.2.18 and 7.2.19 that, under a suitable arrangement, the decomposition matrices $D_{\chi,J}$ and D_χ^J are identical. Comparing this with 7.2.11 and summarizing, we have the following.

<u>Theorem 7.2.20</u> Let $G = (G,B,N,\underline{R},U)$ be a finite group with a split BN-pair of characteristic p and rank ℓ. Suppose that $\chi \in \hat{H}$ and let J be any subset of $\underline{R}$ such that $W_\chi \le W_J$. Then, the decomposition matrix D_χ of the system $(KE_\chi, E_\chi, FE_\chi)$ is identical with the decomposition matrix D_χ^J of the system $(E(KY_\chi^J)$, $E(Y_\chi^J)$, $E(FY_\chi^J))$ where $Y_\chi^J = RL_J[U_{w_J}^-]\beta_\chi$ and L_J is the Levi subgroup of G_J. $\square$

Now suppose that $\chi \in \hat{H}$ with $W_\chi = W_J$ for some $J \subseteq \underline{R}$. Then the K-algebra $KE_{\chi,J}$ (which is isomorphic to KE_χ, by 7.2.6) has K-basis $\{e_\chi^J A_{(w),K}^J,\ w \in W_J\}$ with multiplication relations

$$e_\chi^J A_{(w),K}^J e_\chi^J A_{(v),K}^J = e_\chi^J A_{(v)(w),K}^J \quad \text{if} \quad \ell(vw) = \ell(v) + \ell(w),$$

7.2.21 and

$$e_\chi^J A_{(w_i),K}^{J\,2} = \chi((w_i)^2)\ |U_i|\ e_\chi^J + (\sum_{x \in U_i^*} \chi(h_i(x)))e_\chi^J A_{(w_i),K}^J$$

for all $w,v \in W_J$ and all $w_i \in J$. In particular, if $J = P(\chi)$ then, since $h_i(x)$ and $(w_i)^2$ are in H_i (see 1.2.4), for all $w_i \in J$, the relations 7.2.21 become

7.2.22
$$e_\chi^J A_{(w),K}^J\, e_\chi^J A_{(v),K}^J = e_\chi^J\, A_{(v)(w),K}^J \qquad \text{if } \ell(vw) = \ell(v) + \ell(w)$$
$$e_\chi^J A_{(w_i),K}^{J\,2} = |U_i|\ e_\chi^J + (|U_i| - 1)e_\chi^J\, A_{(w_i),K}^J$$

for all $w,v \in W_J$ and all $w_i \in J$.

Remark Although the case $W_\chi = W_{P(\chi)}$ $(\chi \in \tilde{H})$ is not likely to happen in general (see [HL]), there are some cases in which one can pick a representative λ of the W-orbit (χ) for which this property is valid. However we will see some of these cases in §7.5.

The Case $\chi = 1$:

Now we consider the special case when $\chi = 1$. In this case $Y_1 = RG[B]$ and the R-order E_1 $(= e_1 E(Y_1)e_1)$ has R-basis $\{A_w := e_1 A_w \mid w \in W\}$. The basis element A_w $(w \in W)$ can be identified with the endomorphism of Y_1 which takes $[B] \to [BwB]$.

The R-order E_1 is generated as an R-algebra by the set $\{A_{w_i} \mid w_i \in \underline{R}\}$. The multiplication relations in E_1 can be deduced from 7.2.22 as follows

$$A_w A_v = A_{vw} \quad \text{if } \ell(vw) = \ell(v) + \ell(w) \text{ , } v,w \in W \text{ ,}$$

7.2.23 and

$$A_{w_i}^2 = |U_i| + (|U_i| - 1)A_{w_i} \text{ , for all } w_i \in \underline{R} \text{ .}$$

Consider the decomposition matrix D_1 of the system (KE_1, E_1, FE_1) . Let $\{\theta_i, i \in I\}$ be the set of irreducible characters of G which appear in 1_B^G . It follows that the simple KE_1-modules (and hence the rows of D_1) are indexed by the set I (see [CRII], Theorem 11.25(ii)). Since $P(1) = \underline{R}$, the simple FE_1-modules (hence the columns of D_1) are indexed by all subsets $J \subseteq \underline{R}$.

For $i \in I$ and $J \subseteq \underline{R}$, let $d_{i,J}$ be the (i,J)-entry of D_1 .

By theorem 7.2.3

7.2.24 $\quad d_{i,J} = \langle\theta_i, \eta_J\rangle_G \quad$ for all $i \in I$, $J \subseteq \underline{R}$,

where η_J is the K-character afforded by the KG-module KY(1,J) .
From 6.0.2, we have

7.2.25 $\quad \eta_J = \sum_{J \subseteq S \subseteq \underline{R}} (-1)^{|S\setminus J|} 1_{G_S}^G \quad$ for all $J \subseteq \underline{R}$.

For the rest of this section we will assume that the group $G = G(q)$ is a member of a system L of finite groups with BN-pairs according to the following definition.

<u>Definition</u> (Curtis, Iwahori, Kilmoyer [CIK]): Let $(W,\underline{R})$ be a finite Coxeter system. A <u>system L of finite groups with BN-pairs of type $(W,\underline{R})$</u> consists of the following data:

(i) An infinite set CP of prime powers {q} called characteristic powers.

(ii) For each $q \in CP$, there exists a finite group $G(q) \in L$ with a BN-pair, with Weyl group $(W,\underline{R})$.

(iii) There exists a set of positive integers $\{c_i\}_{w_i \in \underline{R}}$ such that for each $q \in CP$, the index parameters of $G(q)$ are given by

$$|B(q) : B(q) \cap {}^{w_i}B(q)| = q^{c_i} \quad , \text{ for all } w_i \in \underline{R} ,$$

where $B(q)$ is a standard Borel subgroup of $G(q)$.

<u>Examples</u>: (i) Fix n and let $L = \{GL(n, \mathbf{F}_q)\}$, where $\{\mathbb{F}_q\}$ ranges over all finite fields. Then L is a system of finite groups with

BN-pair. CP is the set of all prime powers, $c_i = 1$ for all $w_i \in \underline{R}$, and for all $q \leq CP$,

$$G(q) = GL(n, \mathbb{F}_q) .$$

(ii) Each of the families of finite Chevalley groups, and twisted Chevalley groups (see Carter [CR1]) forms a system of BN-pairs.

<u>The Hecke Algebra $H_K(q)$</u> : Let $G = G(q)$ be a member of a system of finite groups with BN-pairs of type $(W,\underline{R})$, where q is a power of the prime $p > 0$. Let $B(q)$ denote a standard Borel subgroup of $G(q)$. Define, in the group algebra $KG(q)$, the idempotent $b(q) = |B(q)|^{-1} \sum_{b \in B(q)} b$ and the left ideal $V(q) = KG(q).b(q)$. $V(q)$ is a left $KG(q)$-module affording the character $1_{B(q)}^{G(q)}$.

<u>Definition 7.2.26</u> (Iwahori [I]): The Hecke algebra $H_K(q)$ is defined as a subalgebra of $KG(q)$ by

$$H_K(q) = b(q)\ KG(q)\ b(q) .$$

<u>Definition 7.2.27</u> For all $w \in W$, define

$$\mathrm{ind}_{B(q)} w = |B(q) : B(q) \cap B(q)^w| .$$

The Hecke algebra $H_K(q)$ is isomorphic to the opposite ring $(\mathrm{End}_{KG(q)}(KG(q).b(q)))^{op}$, where $\mathrm{End}_{KG(q)}(KG(q).b(q))$ is viewed as an algebra of left operators on $KG(q).b(q)$ (see [CRII], pp.281-282). The simple $H_K(q)$-modules are in 1-1-correspondence with the set of irreducible characters of $G(q)$ which appear in $1_{B(q)}^{G(q)}$ ([CRII], Thm. 11.25(ii)).

The structure of the Hecke algebra $H_K(q)$ has been determined by Iwahori [I] and Matsumoto [HM] as follows:

Theorem 7.2.28 (Iwahori, Matsumoto): $H_K(q)$ has K-basis $\{a_w | w \in W\}$, where $a_w = \mathrm{ind}_{B(q)}^{}w \cdot b(q)(w)b(q)$, for all $w \in W$. $a_1 = b(q)$ is the identity element of $H_K(q)$. For any $1 \neq w \in W$, if $w = w_{i_1} \dots w_{i_s}$ is a reduced expression for w then $a_w = a_{w_{i_1}} \dots a_{w_{i_s}}$. $H_K(q)$ is generated, as K-algebra with identity element a_1, by $\{a_{w_i} \mid w_i \in \underline{R}\}$ and the following are defining relations for $H_K(q)$ in terms of those generators:

$$a_w a_v = a_{wv} \qquad \text{if } \ell(wv) = \ell(w) + \ell(v) ,$$

$$a_{w_i}^2 = q^{c_i} a_1 + (q^{c_i} - 1)a_{w_i} ,$$

for all $w,v \in W$ and $w_i \in \underline{R}$. □

The Generic Algebra

Let $A = Q[u_i ; w_i \in \underline{R}]$ denote the polynomial ring over the rational field Q with generators $\{u_i\}_{w_i \, \underline{R}}$, such that $u_i = u_j$ whenever w_i and w_j are conjugate in W.

Definition 7.2.29 (J. Tits): The generic algebra $H_A(u)$ of the Coxeter system $(W,\underline{R})$ is the associative algebra over the ring A, with identity T_1 and basis $\{T_w \mid w \in W\}$ satisfying:

$$T_w T_v = T_{wv} \qquad , \quad \text{if } \ell(wv) = \ell(w) + \ell(v)$$

$$T_{w_i}^2 = u_i + (u_i - 1)T_{w_i}$$

for all $v,w \; W$ and $w_i \in \underline{R}$.

A <u>specialization</u> is a ring homomorphism $f : A \to \sigma$, with σ a field. σ can be regarded as (σ,A)-bimodule.

<u>Definition</u> Let $H_{f,\sigma} = \sigma \otimes_A H_A(u)$. H_f is called a <u>specialized algebra</u> of $H_A(u)$. It is a σ-algebra with σ-basis $\{1 \otimes T_w = T_{w,f} \mid w \in W\}$ and the structure constants of $H_{f,\sigma}$ are obtained by applying f to the structure constants of the generic algebra $H_A(u)$.

<u>Note</u>: Any specialization $f : A \to \sigma$ induces a ring epimorphism $f : H_A(u) \to H_{f,\sigma}$, such that $f(T_w) := T_{w,f}$ for all $w \in W$.

<u>Examples</u>: 7.2.30 (i) Let $f_q : A \to K$ (K is the field of characteristic 0 in the p-modular system (K,R,F)) be the map given by $f_q(u_i) := q^{c_i}$, $1 \leq i \leq \ell$. Then $H_{f_q,K} \cong H_K(q)$ as K-algebras (see 7.2.27).

(ii) Let $f_1 : A \to K$ be the map given by $f_1(u_i) = 1$, $1 \leq i \leq \ell$. Then $H_{f_1,K} \cong KW$ as K-algebras.

<u>Notations</u>: For the remainder of the section we will fix the following notations:

$\tilde{A} := Q\,(u_i; w_i \in \underline{R})$, the quotient field of A.

L = finite extension of $\tilde{A}$ which is a splitting field for $H_K(u)$.

I_1 = integral closure of A in K.

Any specialization $f : A \to K$ can be extended to a homomorphism $f^* : I_1 \to K$ (see for example [CRII], lemma 68.16).

The specializations f_1 and f_q, defined in 7.2.30, have been used in [CIK] to parametrize the irreducible characters of $G(q)$ which appear in $1_{B(q)}^{G(q)}$. The following theorem given this parametrization in terms of the irreducible characters of W.

<u>Theorem 7.2.30</u> (Curtis, Iwahori, Kilmoyer [CIK]): Let $G(q) \in L$. There exists a bijection $\phi \to \phi_0$ from the set of irreducible K-characters of W to the set

$$\{\phi_0 | \phi_0 \text{ is irreducible character of } G(q) , <\phi_0, 1_{B(q)}^{G(q)}>_{G(q)} \neq 0\} .$$

This bijection depends only on the choices of the extensions f_1^* and f_q^* , and has the following property. For each $J \subseteq \underline{R}$, let W_J and $G_J(q)$ be the corresponding parabolic subgroups of W and $G(q)$, respectively. Then

$$<\phi_0, 1_{G_J(q)}^{G(q)}>_{G(q)} = <\phi, 1_{W_J}^{W}>_W .$$

In particular,

$$<\phi_0, 1_{B(q)}^{G(q)}> = \text{degree of } \phi . \qquad \square$$

We now go back to the decomposition matrix D_1 of the system (KE_1, E_1, FE_1) (with the assumption that $G = G(q) \in L$) . From 7.2.23 and 7.2.28, it is clear that $KE_1 \cong (H_K(q))^{op}$, the opposite ring of $H_K(q)$. Let $\{\theta_i, i \in I\}$ be the set of irreducible characters of $G(q)$ which appear in $1_{B(q)}^{G(q)}$. By theorem 7.2.30, we may use set I to index the full set of irreducible K-characters of W . Let $\{\xi_i, i \in I\}$ be the full set of irreducible K-characters of W , and suppose that the correspondence, given by theorem 7.2.30, takes ξ_i to θ_i $(i \in I)$.

<u>Proposition 7.2.31</u> For every $i \in I$ and $J \subseteq \underline{R}$,

$$d_{i,J} = <\xi_i, \omega_J>_W ,$$

where $\quad \omega_J := \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|}\, 1_{W_S}^{W}$.

Proof By 7.2.24, we have

$$
\begin{aligned}
d_{i,J} &= \langle \theta_i, \eta_J\rangle_{G(q)} \\
&= \langle \theta_i, \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|}\, 1_{G_S(q)}^{G(q)}\rangle_{G(q)} \\
&= \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|} \langle \theta_i , 1_{G_S(q)}^{G(q)}\rangle_{G(q)} \\
&= \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|} \langle \xi_i, 1_{W_S}^{W}\rangle_W \qquad \text{by 7.2.30} \\
&= \langle \xi_i, \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|}\, 1_{W_S}^{W}\rangle_W \\
&= \langle \xi_i, \omega_J\rangle_W \ . \qquad \square
\end{aligned}
$$

§7.3 Some Contravariant Forms On Y_X

Let $G = (G,B,N,\underline{R},U)$ be a finite group with split BN-pair of characteristic p. We assume that G has an involutory anti-automorphism $\theta: G \to G$ (i.e. $\theta(g_1g_2) = \theta(g_2)\theta(g_1)$ for all $g_1, g_2 \in G$ and $\theta^2 = id_G$) with the following axioms:

(i) $\theta(h) = h$ for all $h \in H$.

(ii) $\theta(X_\alpha) = X_{-\alpha}$ for all $\alpha \in \Phi$, when X_α is the root subgroup of G associated with the root $\alpha \in \Phi$ (see §1.2).

(iii) $\theta(N) = N$, and θ induces a map on $W = N/H$ such that $\theta(n)H = n^{-1}H$ for all $n \in N$.

Since $U = \prod_{\alpha\in\Phi^+} X_\alpha$, it follows from (ii) that $\theta(U) = w_0^{-1}Uw_0 = U^-$.

If $G = \langle x_\alpha(t), \alpha \in \Phi, t \in k\rangle$ is a finite Chevalley group defined over a finite field k of characteristic p, then G, with its standard split BN-pair (see [RS1], p.35), has such θ. In fact θ, in this case, is given by

$$\theta(x_\alpha(t)) := x_{-\alpha}(t) \qquad \alpha \in \Phi\,,\ t \in k\,.$$

Definition 7.3.1 Let V and X be two RG-lattices. An R-bilinear form $\beta : V \times X \to R$ is called contravariant (or θ-contravariant) if

$$\beta(gv,x) = \beta(v,\theta(g)x)$$

for all $v \in V$, $x \in X$, and $g \in G$.

If $V \in \operatorname{mod} RG$, denote by V^θ the R-module $\operatorname{Hom}_R(V,R)$, regarded as left RG-module by the rule

7.3.2 $\qquad (gf)(v) := f(\theta(g)v)\,,$

for all $f \in V^\theta$, $g \in G$ and $v \in V$. Note that since V is a free R-module, $\operatorname{Hom}_R(V,R)$ is also a free R-module ([SL], p.343), hence V^θ is an RG-lattice.

Remark 7.3.3 The rule 7.3.2 gives a left G-action on V^θ, since θ is an anti-automorphism. If we replace $\theta(g)$ in 7.3.2 by g^{-1}, we get the usual "dual" or "contragredient" RG-module V^* (see [CRII], p.245). Note also that $V \cong (V^\theta)^\theta$ under the RG-isomorphism $v \to a_v$, $v \in V$, where $a_v \in (V^\theta)^\theta$ is given by

$$a_v(f) := f(v) \quad \text{for all } f \in V^\theta\,.$$

Lemma 7.3.4 Let V , $X \in \text{mod}^{\text{O}}RG$ and let $\beta : V \times X \to R$ be any R-bilinear form. Let $f_\beta : V \to X^\theta$ be the corresponding R-map given by

$$f_\beta(v)(x) := \beta(v,x) ,$$

for all $v \in V$, and all $x \in X$. Then β is contravariant if and only if f_β is an RG-homomorphism.

Proof Suppose β is contravariant. For a given $g \in G$ and $v \in V$ we have

$$\begin{aligned} f_\beta(gv)(x) &= \beta(gv,x) \\ &= \beta(v,\theta(g)x) \\ &= f_\beta(v)(\theta(g)x) \\ &= (g\, f_\beta)(v)(x) \qquad \text{by 7.3.2,} \end{aligned}$$

for all $x \in X$. Hence $f_\beta(gv) = gf_\beta(v)$, for all $g \in G$ and all $v \in V$, and so f_β is an RG-map. Conversely if f_β is an RG-map then, given $g \in G$, $v \in V$, and $x \in X$, we have

$$\begin{aligned} \beta(gv,x) = f_\beta(gv,x) &= f_\beta(gv)(x) \\ &= gf_\beta(v)(x) \qquad \text{since } f_\beta \text{ is an RG-map} \\ &= f_\beta(v)(\theta(g)x) \quad \text{by 7.3.2} \\ &= \beta(v,\theta(g)x) . \end{aligned}$$

Hence β is contravariant. □

If $\{v_1,\ldots,v_r\}$ and $\{x_1,\ldots,x_t\}$ are R-bases for V and X , respectively, then the contravariant form $\beta : V \times X \to R$ of 7.3.4 is completely determined by the values $\beta(v_i,x_j)$, $1 \le i \le r$, $1 \le j \le t$. Let $M(\beta)$ denote the $r \times t$ matrix whose (i,j)-entry is $\beta(v_i,x_j)$.

Definition 7.3.5 Let V , X $\mathrm{mod}^{0}RG$ and let $\beta : V \times X \to R$ be an R-bilinear form. Suppose $\operatorname{rank} V = \operatorname{rank} X = t$; say. Then β is said to be non-singular if the $t \times t$ matrix $M(\beta)$ is invertible in R , equivalently if the map $f_\beta : V \to X^\theta$ is an isomorphism of R-modules.

In this section we will show that the set $\{Y_\chi, \chi \in \operatorname{Hom}(H,K^\times)\}$ is closed under the θ-duality. We will also show that the simple FG-modules $M(\chi,J)$ are self θ-dual.

Let $\chi \in \tilde{H} = \operatorname{Hom}(H,K^\times)$. If S is a transversal of $\{gB \; ; \; g \in G\}$ then it is clear that $\{s[U]\beta_\chi \; ; \; s \in S\}$ forms an R-basis for the RG-lattice Y_χ $(= RG[U]\beta_\chi)$.

Lemma 7.3.6 Let S be a transversal of the set $\{gB \; ; \; g \in G\}$. Then $\{\theta(s)^{-1}(w_0) \; ; \; s \in S\}$ is also a transversal of $\{gB \; ; \; g \in G\}$.

Proof If $g \in G$ then $g = sb$ for a unique $s \in S$ and a unique $b \in B$, and $\theta(g^{-1}) = \theta(b^{-1}s^{-1}) = \theta(s^{-1})\,(b^{-1})$. Since $\theta(B) = w_0 B w_0^{-1}$, $\theta(b^{-1}) = (w_0)b'(w_0)^{-1}$ for some $b' \in B$, and so $\theta(g^{-1}) = \theta(s^{-1})(w_0)b'(w_0)^{-1}$, hence $\theta(g^{-1})(w_0) = \theta(s^{-1})(w_0)b'$. Since the map $g \to \theta(g^{-1})(w_0)$ is a bijection of the elements of G , it follows that $sb \to \theta(s^{-1})(w_0)b'$ is also a bijection of the elements of G . Hence every element $g \in G$ has the form

$$\theta(s^{-1})(w_0)b' \qquad \text{for a unique } s \in S$$

and a unique $b' \in B$. Therefore $\{\theta(s^{-1})(w_0) \; ; \; s \in S\}$ is a transversal of $\{gB \; ; \; g \in G\}$. $\square$

Proposition 7.3.7 There is a non-singular contravariant form $< \, , \, > : Y_\chi \times Y_{w_0\chi} \to R$ given by

$$\text{7.3.8} \qquad <g[U]\beta_\chi \, , \, \theta(g')^{-1}(w_0)[U]\beta_{w_0\chi} := \begin{cases} 0 & \text{if } g'^{-1}g \notin B \\ \chi(g'^{-1}g) & \text{if } g'^{-1}g \in B \end{cases}$$

for all $g \, , \, g' \in G$.

Proof (i) First we need to show that the form $< \, , \, >$ is well-defined; that is if we replace g and g' in 7.3.8 by gb and $g'b'$, respectively, where $b \, , \, b' \in B$, then the right hand side of 7.3.8 will have the appropriate value. We have

$$\begin{aligned} &<gb[U]\beta_\chi \, , \, \theta(g'b')^{-1}(w_0)[U]\beta_{w_0\chi}> \\ &= <gb\beta_\chi[U], \theta(b'^{-1}g'^{-1})(w_0)[U]\beta_{w_0\chi}> \\ &= \chi(b)<g\beta_\chi[U], \, \theta(g'^{-1})\theta(b'^{-1})(w_0)[U]\beta_{w_0\chi}> \\ &= \chi(b)<g\beta_\chi[U], \theta(g'^{-1})(w_0)(w_0)^{-1}\theta(b'^{-1})(w_0)[U]\beta_{w_0\chi}> \\ &= \chi(b)w_0\chi((w_0)^{-1}\theta(b')^{-1}(w_0))<g[U]\beta_\chi, \theta(g')^{-1}(w_0)[U]\beta_{w_0\chi}> \; . \end{aligned}$$

If we write $b' = h'u'$ for some $h' \in H$ and $u' \in U$, then

$$\begin{aligned} (w_0)^{-1}\theta(b')^{-1}(w_0) &= (w_0)^{-1}\theta(h'u')^{-1}(w_0) \\ &= (w_0)^{-1}\theta(h')^{-1}\theta(u')^{-1}(w_0) \\ &= (w_0)^{-1}\theta(h')^{-1}(w_0)(w_0)^{-1}\theta(u')^{-1}(w_0) \in UH \; . \end{aligned}$$

Since $w_o\chi\Big|_U = 1_U$, we have

$$w_o\chi((w_o)^{-1}\theta(b')^{-1}(w_o)) = w_o\chi((w_o)^{-1}\theta(h')^{-1}(w_o))$$

$$= w_o\chi((w_o)^{-1}h'^{-1}(w_o))$$

$$= \chi(h')^{-1}$$

$$= \chi(u'^{-1}h'^{-1}) = \chi(b')^{-1} .$$

Therefore,

$$<gb[U]\beta_\chi \, , \, \theta(g'b')^{-1}(w_o)[U]\beta_{w_o\chi}>$$

$$= \chi(b)\chi(b')^{-1}<g[U]\beta_\chi,\theta(g'^{-1})(w_o)[U]\beta_{w_o\chi}>$$

$$= \begin{cases} \chi(bb'^{-1})\chi(gg'^{-1}) & \text{if } gg'^{-1} \in B \\ 0 & \text{if } gg'^{-1} \notin B , \end{cases}$$

which shows that $<\ ,\ >$ is well-defined.

(ii) We show that $<\ ,\ >$ is contravariant. If $g,g',g_1 \in G$ then

$$<g_1g[U]\beta_\chi,\theta(g')^{-1}(w_o)[U]\beta_{w_o\chi}>$$

$$= \begin{cases} \chi(g'^{-1}g_1g) & \text{if } g'^{-1}g_1g \in B \\ 0 & \text{if } g'^{-1}g_1g \notin B \end{cases}$$

$$= <g[U]\beta_\chi,\theta(g_1^{-1}g')^{-1}(w_o)[U]\beta_{w_o\chi}>$$

$$= <g[U]\beta_\chi,\theta(g_1)\theta(g'^{-1})(w_o)[U]\beta_{w_o\chi}> .$$

Hence $<\ ,\ >$ is θ-contravariant.

(iii) Non-singularity. Let S be a transversal of the set $\{gB \; ; \; g \in G\}$. By 7.3.6 $\{\theta(s)^{-1}(w_0)[U]\beta_{w_0\chi}, \; s \in S\}$ is an R-basis for $Y_{w_0\chi}$. For $s,s' \in S$ we have

$$<s[U]\beta_\chi, \theta(s')^{-1}(w_0)[U]\beta_{w_0\chi}> = \begin{cases} 0 & \text{if } s'^{-1} \notin B \\ \chi(s'^{-1}s) & \text{if } s'^{-1}s \in B . \end{cases}$$

But $s'^{-1}s \in B \iff sB = s'B \iff s = s'$. Therefore we have

$$<s[U]\beta_\chi, \theta(s')^{-1}(w_0)[U]\beta_{w_0\chi}> = \delta_{s,s'} \quad ,$$

for all $s,s' \in S$, and so the $|S| \times |S|$ matrix $M(< , >)$ is invertible hence $< , >$ is non-singular. $\square$

Remarks 7.3.9 (i) It follows from 7.3.7 that there exists an RG-isomorphism $f_{< , >} : Y_\chi \to Y^\theta_{w_0\chi}$, induced from the contravariant form $< , >$, given by

$$f_{< , >}(x)(y) := <x,y>$$

for all $x \in Y_\chi$ and $y \in Y_{w_0\chi}$.

(ii) If $k \in \{K,F\}$ then the contravariant form $< , > : Y_\chi \times Y_{w_0\chi} \to R$, defined in 7.3.7, will induce a contravariant form $< , >_k : kY_\chi \times kY_{w_0\chi} \to k$. It is clear that the form $< , >_k$ is non-singular.

Let (χ,J) be an admissible G-pair and let $M(\chi,J)$ be the corresponding simple FG-module. In ([C2], Theorem 6.15), Curtis shows that $M(\chi,J)$ is completely determined by its unique B-line (see 4.2.5) and the parabolic subgroup G_J which is the full stabiliser of that line.

We will assume that the Coset representatives $\{(w_i), w_i \in \underline{R}\}$ are chosen such that $(w_i) \in \langle U_i, U_{-i}\rangle$ for all $w_i \in \underline{R}$. Such choice is possible by 1.2.4(i).

Theorem 7.3.10 $M(\chi,J)^{\theta} \cong M(\chi,J)$ as FG-modules, for all admissible G-pairs (χ,J) .

Proof Let $M = M(\chi,J)$. It is clear, since $(M^{\theta})^{\theta} = M$, that M^{θ} is simple FG-module. Let F.m be the unique B-line of M . From ([C2], Theorem 4.3), we have

$$\begin{aligned} M &= FG.m \\ &= (FU^{-}).m \\ &= F.m \oplus \underline{r}(FU^{-})m \quad , \qquad (7.3.11) \end{aligned}$$

where $\underline{r}(FU^{-})$ is the radical of FU^{-} . Define the F-map $\mu : M \to F$ as follows: If $m' \in M$ then $\mu(m')$ is the coefficient of m in the decomposition of m' according to 7.3.11, that is

$$m' = \mu(m')m + x_1 \quad , \quad x_1 \in \underline{r}(FU^{-})m \quad .$$

For every $u \in U$ we have

$$\begin{aligned} \theta(u)m' &= \mu(m')\theta(u)m + \theta(u)x_1 \\ &= \mu(m')((\theta(u)-1)m + m) + \theta(u)x_1 \\ &= \mu(m')m + \mu(m')(\theta(u)-1)m + \theta(u)x_1 \ . \end{aligned}$$

Since $\theta(u) \in U^{-}$ and U^{-} is a p-group, it follows (see [CRII], Thm. 5.24) that $(\theta(u)-1) \in \underline{r}(FU^{-})$. It is also clear that $\theta(u)x_1 \in \underline{r}(FU^{-})m$.

Hence

$$\theta(u)m' = \mu(m')m + x_2 \quad , \quad x_2 \in \underline{r}(FU^{-})m$$

and so $u\mu = \mu$ for all $u \in U$. If $h \in H$ then, since $hm = \chi(h)m$, we have

$$hm' = \chi(h)\mu(m')m + hx_1$$

and $hx_1 \in \underline{r}(FU^-)m$, since H normalizes U^-. Therefore $h\mu = \chi(h)$ for all $h \in H$. It follows that $F.\mu$ is the unique B-line of M^θ and that $M^\theta \cong M(\chi,T)$ for some $T \subseteq P(\chi)$.

To complete the proof we need to show that G_J is the full stabilizer of $F.\mu$. Let $w_i \in J$ and let

$$m' = \mu(m')m + x_1 \quad , \quad x_1 \in \underline{r}(FU^-)m$$

be any element of M. We have

7.3.12 $\qquad \theta((w_i))m' = \mu(m')\theta((w_i))m + \theta((w_i))x_1$.

Since $(w_i) \in \langle U_i, U_{-i}\rangle$, it follows from axiom (ii) of θ that $\theta((w_i)) \in \langle U_i, U_{-i}\rangle$ (note that $U_i = U_{\alpha_i}$). By 1.2.4(3), we then have $\theta((w_i)) = (w_i)h$ for some $h \in H_i$. Since G_J is the full stabilizer of $F.m$ and $\chi\big|_{H_i} = 1$, it follows from 7.3.12 that

$$\begin{aligned}\theta((w_i))m' &= \mu(m')\chi(h)m + (w_i)hx_1 \\ &= \mu(m')m + (w_i)hx_1 .\end{aligned}$$

It will then follow that $(w_i)\mu = \mu$ if we show that

7.3.13 $\qquad (w_i)h\, \underline{r}(FU^-)m \subseteq \underline{r}(FU^-)m$.

Since H normalises U^- and $hm = \chi(h)m$, we have

7.3.14 $\qquad (w_i)h\, \underline{r}(FU^-)m = (w_i)\underline{r}(FU^-)m$.

To show that $(w_i)\underline{r}(FU^-)m \subseteq \underline{r}(FU^-)m$, consider the decomposition

$$U^- = (U^-)^+_{w_i}(U^-)^-_{w_i} \quad ,$$

which follows from 1.2.8, where $(U^-)^+_{w_i} = U^- \cap (U^-)^{w_i}$ and $(U^-)^-_{w_i} = U^- \cap (U^-)^{w_o w_i} = U^- \cap U^{w_i}$. Since U^- is a p-group, $\underline{r}(FU^-)$ is spanned by the elements $v-1$, $v \in U^-$. If $v \in U^-$, write $v = v_1 v_2$, where $v_1 \in (U^-)^+_{w_i}$ and $v_2 \in (U^-)^-_{w_i}$. Then $(v-1) = (v_1v_2-1) = v_1(v_2-1) + (v_1-1)$, hence

$$\begin{aligned}(w_i)(v-1)(w_i)^{-1} &= (w_i)(v_1(v_2-1) + (v_1-1))(w_i)^{-1} \\ &= z_1(z_2-1) + (z_1-1) \;,\end{aligned}$$

where $z_1 = (w_i)v_1(w_i)^{-1}$ and $z_2 = (w_i)v_2(w_i)^{-1}$. But $z_2 \in (w_i)(U^-)^-_{w_i}(w_i)^{-1} = (U^-)^{w_i} \cap U \leq U$, and so $(z_2-1)m = 0$, hence

$$\begin{aligned}(w_i)(v-1)(w_i)^{-1}m &= z_1(z_2-1)m + (z_1-1)m \\ &= (z_1-1)m \in \underline{r}(FU^-)m \;.\end{aligned}$$

Therefore (since $(w_i)m = m$),

$$\begin{aligned}(w_i)\underline{r}(FU^-)m &= (w_i)\underline{r}(FU^-)(w_i)^{-1}(w_i)m \\ &= (w_i)\underline{r}(FU^-)(w_i)^{-1}m \subseteq \underline{r}(FU^-)m \quad ,\end{aligned}$$

which proves 7.3.13 and so $(w_i)\mu = \mu$ for all $w_i \in J$. Since G_T is the full stabilizer of $F.\mu$, $J \subseteq T$.

On the other hand, since $(M^\theta)^\theta = M$, we have $M(\chi,T)^\theta \cong M(\chi,J)$ and so by reversing the above argument we get $T \subseteq J$. Hence $T = J$ and so $M(\chi,J)^\theta = M(\chi,J)$. This completes the proof of 7.3.10. □

Remark 7.3.16 The functor $(\ ,R) : \text{mod}^{o}RG \to \text{mod}\ R$ is R-linear, hence it commutes with direct sum of RG-lattices. Note also that, since $k \underset{R}{\otimes} (X,R) \cong (kX,k)$ (X is any RG-lattice and $k \in \{K,F\}$), it follows that θ-dualizing commutes with the functor $k \underset{R}{\otimes} : \text{mod}^{o}RG \to \text{mod}\ kG$.

For every $J \subseteq \underline{R}$, let ${}^{w_o}J = w_o J w_o^{-1}$. ${}^{w_o}J \subseteq \underline{R}$ (see [HS], p.38), and if (χ,J) is an admissible G-pair then so is $(w_o\chi, {}^{w_o}J)$ and the mapping $(\chi,J) \to (w_o\chi, {}^{w_o}J)$ is a bijection.

Recall that for every $\chi \in \hat{H}$, we have $Y_\chi = \sum^{\oplus}_{J \subseteq P(\chi)} Y(\chi,J)$. By 7.3.9(i), $Y^{\theta}_{w_o\chi} \cong Y_\chi$ and so, as the θ-dual functor commutes with direct sums of RG-lattices, we then have

$$Y_\chi \cong \sum^{\oplus}_{S \subseteq P(w_o\chi)} Y(w_o\chi,S)^{\theta} \quad .$$

Using Krull-Schmidt theorem for RG-lattices (see [CRII], p.620), we then have

7.3.17 $\qquad Y(\chi,J) \cong Y(w_o\chi,S)^{\theta}$

for a unique $S \subseteq P(w_o\chi)$.

Proposition 7.3.18 $Y(\chi,J) \cong Y(w_o\chi, {}^{w_o}J)^{\theta}$ as RG-lattices, for all admissible G-pairs (χ,J).

Proof From 7.3.17, we have

7.3.19 $\qquad Y(\chi,J) \cong Y(w_o\chi,S)^{\theta}$

for some $S \subseteq P(w_o\chi)$. By taking the θ-dual of 7.3.19 we get

$$Y(\chi,J)^\theta \cong Y(w_0\chi,S) ,$$

and, by remark 7.3.16, we have $FY(\chi,J)^\theta \cong FY(w_0\chi,S)$. To show that $S = {}^{w_0}J$, it is enough to show that $\mathrm{Soc}(FY(w_0\chi,S)) \cong M(\chi,J)$ (see [HS], theorem 3.10). But since the head of $FY(\chi,J)$ is $M(\chi,J)$ and by 7.3.10, we have

$$\mathrm{Soc}(FY(w_0\chi,S)) \cong \mathrm{Soc}(FY(\chi,J)^\theta) \cong M(\chi,J)^\theta \cong M(\chi,J) .$$

Hence $S = {}^{w_0}J$, and so

$$Y(\chi,J) \cong Y(w_0\chi, {}^{w_0}J)^\theta \quad . \qquad \square$$

Now let $\chi \in \hat{H}$ and let $\{X_i, i \in I\}$ be a set of RG-lattices such that $\{KX_i, i \in I\}$ is the full set of simple KG-components of KY_χ . Since $KY_\chi \cong KY_{w_0\chi}$ (see 5.0.4(ii)), we can identify the KG-components of KY_χ with those of $KY_{w_0\chi}$. On the other hand the RG-isomorphism $f_{<\ ,\ >} : Y_\chi \to Y^\theta_{w_0\chi}$, defined in 7.3.9(i), induces a KG-isomorphism

$$1_K \otimes f_{<\ ,\ >} : KY_\chi \to (KY_{w_0\chi})^\theta \quad .$$

Therefore, for each $i \in I$, $(KX_i)^\theta \cong KX_{i'}$ for a unique $i' \in I$ (note that $(KX_i)^\theta$ is simple since $((KX_i)^\theta)^\theta \cong KX_i$, and the θ-dual functor commutes with direct sums). Since $((KX_i)^\theta)^\theta \cong KX_i$, the map $i \to i'$ defines an involution on the index set I .

Let (χ,J) be an admissible G-pair, and suppose that

7.3.20 $$KY(\chi,J) \cong \sum_{i\in I}^{\oplus} d_{i,(\chi,J)}\, KX_i ,$$

and

7.3.21 $\quad KY(w_0\chi, {}^{w_0}J) \cong \sum_{i\in I}^{\oplus} d_{i,(w_0\chi, {}^{w_0}J)} KX_i$,

where $d_{i,(\chi,J)}$ and $d_{i,(w_0\chi, {}^{w_0}J)} \in \mathbb{Z}_{\geq 0}$.

By applying the θ-dual functor to 7.3.21 we get

7.3.22 $$(KY(w_0\chi, {}^{w_0}J))^\theta \cong \sum_{i\in I} d_{i,(w_0\chi, {}^{w_0}J)} (KX_i)^\theta$$

$$= \sum_{i\in I} d_{i,(w_0\chi, {}^{w_0}J)} KX_{i'} \quad .$$

By 7.3.18 and remark 7.3.16 we have

7.3.23 $$(KY(w_0\chi, {}^{w_0}J))^\theta \cong KY(\chi,J)$$

as KG-modules. Therefore, for each $i \in I$

$$d_{i,(w_0\chi, {}^{w_0}J)} = [KX_{i'} \mid (KY(w_0\chi, {}^{w_0}J))^\theta] \qquad \text{by 7.3.22}$$

$$= [KX_{i'} \mid KY(\chi,J)] \qquad \text{by 7.3.23}$$

7.3.24 $$= d_{i',(\chi,J)} \quad .$$

Using dual argument to the above, we also have

7.3.25 $$d_{i,(\chi,J)} = d_{i',(w_0\chi, {}^{w_0}J)} \quad .$$

Now consider the decomposition matrices D_χ and $D_{w_0\chi}$ of the systems $(KE_\chi , E_\chi , FE_\chi)$ and $(KE_{w_0\chi}, E_{w_0\chi}, FE_{w_0\chi})$, respectively. Since $KY_\chi \cong KY_{w_0\chi}$, the set I indexes the rows of D_χ and $D_{w_0\chi}$.

On the other hand

$$Y_\chi = \sum_{J\subseteq P(\chi)}^{\oplus} Y(\chi,J) \quad \text{and} \quad Y_{w_0\chi} = \sum_{S\subseteq P(w_0\chi)}^{\oplus} Y(w_0\chi,S)$$

with $|P(\chi)| = |P(w_0\chi)|$. Therefore the set of simple FE_χ-modules and the set of simple $FE_{w_0\chi}$-modules are in 1-1 correspondence. Consequently D_χ and $D_{w_0\chi}$ have the same size, namely $|I| \times 2^{|P(\chi)|}$. Moreover, using 7.2.3, it follows from 7.3.24 and 7.3.25 that D_χ and $D_{w_0\chi}$ have the following forms

$$D_\chi = \begin{array}{c} \\ i \\ \\ i' \end{array} \begin{array}{c} J \\ \vdots \\ \left(\begin{array}{c} \ldots d_{i,(\chi,J)} \\ \\ \ldots d_{i',(\chi,J)} \end{array}\right) \end{array} , \qquad D_{w_0\chi} = \begin{array}{c} \\ i \\ \\ i' \end{array} \begin{array}{c} {}^{w_0}J \\ \vdots \\ \left(\begin{array}{c} \ldots d_{i,(w_0\chi,{}^{w_0}J)} \\ \\ \ldots d_{i',(w_0\chi,{}^{w_0}J)} \end{array}\right) \end{array} ,$$

with the i-th row of $D_{w_0\chi}$ $(i \in I)$ identical to the i'-th row of D_χ . Summarizing the above we have

<u>Theorem 7.3.26</u> Let $\chi \in \hat{H}$, and suppose that the set of simple KG-components of KY_χ are indexed by the finite set I . Let $i \to i'$ $(i \in I)$ be the involution on I defined above. Then for every $i \in I$ and every $J \subseteq P(\chi)$,

$$d_{i,(\chi,J)} = d_{i',(w_0\chi,{}^{w_0}J)} \quad \text{and} \quad d_{i',(\chi,J)} = d_{i,(w_0\chi,{}^{w_0}J)} .$$

Consequently, the index set I can be arranged so that D_χ and $D_{w_0\chi}$ are identical. $\square$

§7.4 The Direct Product of Split BN-pairs

In this section we consider the direct product of two finite groups with split BN-pairs. We will show that such product also has a split BN-pair, and will derive some results which are relevant to the subject of chapter 7. In particular, we will prove some results concerning the decomposition numbers $d_{i,(\chi,J)}$ (see 7.2.3). These results will be applied later in §7.5 to the case of the general linear group.

Let $G^{(i)} = (G^{(i)}, B^{(i)}, N^{(i)}, \underline{R}^{(i)}, U^{(i)})$, $i = 1,2$, be a finite group with a split BN-pair of rank ℓ_i and characteristic p , for some prime $p > 0$. Write $H^{(i)} = B^{(i)} \cap N^{(i)}$ and $W^{(i)} = N^{(i)}/H^{(i)}$. Let $G = G^{(1)} \times G^{(2)}$ be the direct product of the groups $G^{(1)}$ and $G^{(2)}$. Every element g of G can be written as a "vector" $g = (g_1,g_2)$, where $g_i \in G^{(i)}$, $i = 1,2$. We will sometimes write $G = (G^{(1)},G^{(2)})$.

<u>Lemma 7.4.1</u> Suppose $G = G^{(1)} \times G^{(2)}$, where $G^{(i)} = (G^{(i)},B^{(i)},N^{(i)}, \underline{R}^{(i)},U^{(i)})$, $i = 1,2$, is a finite group with a split BN-pair of characteristic p , for some prime $p > 0$. Then $B = B^{(1)} \times B^{(2)}$ and $N = N^{(1)} \times N^{(2)}$ form a split BN-pair in G . Moreover, $(W = W^{(1)} \times W^{(2)}$, $\underline{R} = \underline{R}^{(1)} \dot{\cup} \underline{R}^{(2)})$ is the Coxeter system of this BN-pair.

<u>Proof</u> We need to verify the axioms of BN-pairs.

(1) Since $G^{(i)} = \langle B^{(i)}, N^{(i)}\rangle$, $i = 1,2$,

$$\begin{aligned} G = G^{(1)} \times G^{(2)} &= \langle B^{(1)},N^{(1)}\rangle \times \langle B^{(2)},N^{(2)}\rangle \\ &= \langle B^{(1)} \times B^{(2)} , N^{(1)} \times N^{(2)}\rangle \\ &= \langle B , N\rangle \ . \end{aligned}$$

(2) $$B \cap N = B^{(1)} \times B^{(2)} \cap N^{(1)} \times N^{(2)}$$
$$= B^{(1)} \cap N^{(1)} \times B^{(2)} \cap N^{(2)}$$
$$= H^{(1)} \times H^{(2)} \trianglelefteq N^{(1)} \times N^{(2)} ,$$

since $H^{(i)} \trianglelefteq N^{(i)}$; $i = 1,2$. Therefore

$$N/B\cap N = N^{(1)} \times N^{(2)}/H^{(1)} \times H^{(2)} = W^{(1)} \times W^{(2)} .$$

Suppose that $\underline{R}^{(i)} = \{w_1^{(i)},\ldots, w_{\ell_i}^{(i)}\}$; $i = 1,2$. The subgroup $<(w_j^{(1)},1)$; $j = 1,\ldots,\ell_1>$ of $W^{(1)} \times W^{(2)}$ is isomorphic to $W^{(1)}$. So we may identify $W^{(1)}$ with $<(w_j^{(1)},1)$; $j = 1,\ldots,\ell_1>$, and $\underline{R}^{(1)}$ with $\{(w_j^{(1)},1)$; $j = 1,\ldots,\ell_1\}$. Similar identification can be done for $W^{(2)}$ and $\underline{R}^{(2)}$, and in this way we have $\underline{R} = \underline{R}^{(1)} \dot{\cup} \underline{R}^{(2)}$.

(3) Suppose $r \in \underline{R}$ and $w = (w^{(1)},w^{(2)}) \in W^{(1)} \times W^{(2)}$. Since $R = \underline{R}^{(1)} \dot{\cup} \underline{R}^{(2)}$, we may assume that $r \in \underline{R}^{(i)}$, $i = 1,2$. Suppose $r = w_j^{(1)} \in \underline{R}^{(1)}$, then identifying $w_j^{(1)}$ with $(w_j^{(1)},1) \in W^{(1)} \times W^{(2)}$ we have

$$rBw = (w_j^{(1)},1)\ (B^{(1)},B^{(2)})\ (w^{(1)},w^{(2)})$$
$$= (w_j^{(1)}\ B^{(1)}w^{(1)},\ B^{(2)}w^{(2)})$$
$$\subseteq (B^{(1)}w_j^{(1)}w^{(1)}B^{(1)}\ \dot{\cup}\ B^{(1)}w^{(1)}B^{(1)},B^{(2)}w^{(2)}) ,$$

by the BN-pair axioms of $G^{(1)}$.

$$= (B^{(1)}w_j^{(1)}w^{(1)}B^{(1)},B^{(2)}w^{(2)})\ \dot{\cup}\ (B^{(1)}w^{(1)}B^{(1)},B^{(2)}w^{(2)})$$
$$\subseteq (B^{(1)}w_j^{(1)}w^{(1)}B^{(1)},B^{(2)}w^{(2)}B^{(2)})\ \dot{\cup}\ (B^{(1)}w^{(1)}B^{(1)},B^{(2)}w^{(2)}B^{(2)})$$
$$= (B^{(1)},B^{(2)})(w_j^{(1)}w^{(1)},w^{(2)})(B^{(1)},B^{(2)})$$
$$\dot{\cup}\ (B^{(1)},B^{(2)})(w^{(1)},w^{(2)})(B^{(1)},B^{(2)})$$
$$= BrwB\ \dot{\cup}\ BwB .$$

The case when $r \in \underline{R}^{(2)}$ is similar.

(4) For $r \in \underline{R}$, using similar discussion to that in (3) and the BN-pair axioms of the groups $G^{(i)}$; $i = 1,2$, one can easily see that $rBr \nsubseteq B$.

(5) Finally we have

$$B = B^{(1)} \times B^{(2)} = U^{(1)}H^{(1)} \times U^{(2)}H^{(2)}$$
$$= (U^{(1)} \times U^{(2)})\ (H^{(1)} \times H^{(2)}) .$$

Therefore, putting $U = U^{(1)} \times U^{(2)}$ and $H = H^{(1)} \times H^{(2)}$, we then have $U \trianglelefteq B$ and $B = UH$.

It follows from (1)-(5) that

$$(G^{(1)}{\times}G^{(2)},\ B^{(1)}{\times}B^{(2)},\ N^{(1)}{\times}N^{(2)},\ \underline{R}^{(1)} \dot{\cup} R^{(2)},\ U^{(1)}{\times}U^{(2)})$$

form a split BN-pair in $G = G^{(1)} \times G^{(2)}$ of rank $\ell_1 + \ell_2$. It is also clear that this split BN-pair is of the same characteristic as $(G^{(i)},B^{(i)},N^{(i)},\underline{R}^{(i)},U^{(i)})$, $i = 1,2$. This completes the proof of the lemma. □

Let (K,R,F) be a p-modular system such that K (and hence F) is a splitting field for $G^{(i)}$, $i = 1,2$, and all its subgroups.

If $\chi \in \hat{H} = \mathrm{Hom}(H,K^{\times})$, then χ has the form (χ_1,χ_2), where $\chi_i \in \hat{H}^{(i)} = \mathrm{Hom}(H^{(i)},K^{\times})$, $i = 1,2$. The value of χ $(= (\chi_1,\chi_2))$ or $h = (h_1,h_2) \in H$ is given by

7.4.2 $\qquad \chi(h) := \chi_1(h_1)\chi_2(h_2)$.

Any $\chi = (\chi_1,\chi_2) \in \hat{H}$ can be extended to a character χ_B of B by putting

$$\chi_B = (\chi_{1B^{(1)}} , \chi_{2B^{(2)}}) ,$$

where $\chi_{iB^{(i)}}$, $i = 1,2$, is the extension of χ_i to a character of $B^{(i)}$ which has the trivial value on $U^{(i)}$.

Let $\chi = (\chi_1,\chi_2) \in \hat{H}$, and for $i = 1,2$, let $Y^{(i)}_{\chi_i} = RG^{(i)}[U^{(i)}]\beta_{\chi_i}$, where $\beta_{\chi_i} = |H^{(i)}|^{-1} \sum_{h \in H^{(i)}} \chi_i(h^{-1})h$. The R-lattice $Y^{(1)}_{\chi_1} \otimes_R Y^{(2)}_{\chi_2}$ has a structure of left $R(G^{(1)} \times G^{(2)})$-module (see [CRII], §10E), where the $G^{(1)} \times G^{(2)}$-action is given by:

$$(g_1,g_2)(x_1 \otimes x_2) := g_1x_1 \otimes g_2x_2 ,$$

for all $(g_1,g_2) \in G^{(1)} \times G^{(2)}$ and all $x_1 \in Y^{(1)}_{\chi_1}$, $x_2 \in Y^{(2)}_{\chi_2}$.

Let $Y_\chi = RG[U]\beta_\chi$, where $G = G^{(1)} \times G^{(2)}$, $U = U^{(1)} \times U^{(2)}$ and $\beta_\chi = |H|^{-1} \sum_{h \in H} \chi(h^{-1})h$.

<u>Lemma 7.4.3</u> $Y_\chi \cong Y^{(1)}_{\chi_1} \otimes_R Y^{(2)}_{\chi_2}$ as RG-lattices

<u>Proof</u> Let $\Omega : RG \to RG^{(1)} \otimes RG^{(2)}$ be the R-map given by $\Omega((g_1,g_2)) := g_1 \otimes g_2$, for all $(g_1,g_2) \in G$. Ω is clearly an R-isomorphism. It is also an RG-isomorphism where the G-action on $RG^{(1)} \otimes_R RG^{(2)}$ is the one given by

$$(g_1,g_2)(x_1 \otimes x_2) = g_1x_1 \otimes g_2x_2 ,$$

for all $(g_1,g_2) \in G$, $x_1 \in RG^{(1)}$, and $x_2 \in RG^{(2)}$. It is clear that if $g = (g_1,g_2) \in G$, then Ω sends $g[U]\beta_\chi \in RG$ to $g_1[U^{(1)}]\beta_{\chi_1} \otimes g_2[U^{(2)}]\beta_{\chi_2} \in RG^{(1)} \otimes RG^{(2)}$. Therefore

$$\begin{aligned}\Omega(Y_\chi) &= \Omega(RG[U]\beta_\chi) \\ &\cong RG^{(1)}[U^{(1)}]\beta_{\chi_1} \underset{R}{\otimes} RG^{(2)}[U^{(2)}]\beta_{\chi_2} \\ &= Y^{(1)}_{\chi_1} \underset{R}{\otimes} Y^{(2)}_{\chi_2} \quad .\end{aligned}$$

Hence $Y_\chi \cong Y^{(1)}_{\chi_1} \otimes Y^{(2)}_{\chi_2}$ as RG-lattices. □

It follows from 6.0.7 that we have a decomposition

7.4.4 $$Y_\chi \cong \sum_{J\subseteq P(\chi)}^{\oplus} Y(\chi,J) ,$$

of Y_χ into mutually non-isomorphic indecomposable RG-lattices. Let $w_o^{(i)}$, $i = 1,2$, be the unique element of $W^{(i)}$ of maximal length. Then it is clear that $w_o = (w_o^{(1)},w_o^{(2)}) \in W^{(1)} \times W^{(2)}$ is the unique element of $W^{(1)} \times W^{(2)}$ of maximal length.

Let $w_j \in \underline{R}$ $(= \underline{R}^{(1)} \dot{\cup} \underline{R}^{(2)})$, and suppose that $w_j = w_j^{(1)} \in \underline{R}^{(1)}$, $1 \le j \le \ell_1$. Then, identifying $w_j^{(1)}$ with $(w_j^{(1)},1) \in W^{(1)} \times W^{(2)}$, we have $w_ow_j = (w_o^{(1)}w_j^{(1)},w_o^{(2)})$. Hence

$$\begin{aligned}U_j = U^-_{w_j} = U \cap U^{w_ow_j} &= (U^{(1)}\cap(U^{(1)})^{w_o^{(1)}w_j^{(1)}} , U^{(2)} \cap (U^{(2)})^{w_o^{(2)}}) \\ &= (U_j^{(1)}, 1) .\end{aligned}$$

Therefore

$$\begin{aligned}U_{-j} = U_j^{w_j} &= ((U_j^{(1)})^{w_j^{(1)}} , 1) \\ &= (U_{-j}^{(1)} , 1) ,\end{aligned}$$

and so

7.4.5 $$H_j = H \cap \langle U_j, U_{-j} \rangle$$
$$= (H^{(1)}, H^{(2)}) \cap (\langle U_j^{(1)}, U_{-j}^{(1)} \rangle, 1)$$
$$= (H_j^{(1)}, 1) .$$

Similarly, if $w_j = w_j^{(2)} \in \underline{R}^{(2)}$, $1 \leq j \leq \ell_2$, then

7.4.6 $$H_j = (1, H_j^{(2)}) .$$

Lemma 7.4.7 Let $\chi = (\chi_1, \chi_2) \in \hat{H}$, where $\chi_i \in \hat{H}^{(i)}$, $i = 1,2$. Then

(i) $P(\chi) = P(\chi_1) \dot{\cup} P(\chi_2)$, and

(ii) (χ, J) is an admissible G-pair if and only if $J = J_1 \dot{\cup} J_2$, say $J_1 \subseteq \underline{R}^{(1)}$, $J_2 \subseteq \underline{R}^{(2)}$, and (χ_i, J_i) is an admissible $G^{(i)}$-pair, $i = 1,2$.

Proof Both (i) and (ii) follows from 7.4.5 and 7.4.6. □

Let $\chi = (\chi_1, \chi_2) \in \hat{H}$. Since $Y_\chi \cong Y_{\chi_1} \otimes_R Y_{\chi_2}$ as RG-lattices, it follows that $E(Y_\chi) = E(Y_{\chi_1}) \otimes E(Y_{\chi_2})$ (see [CRII], Lemma 10.37). To define an explicit algebra isomorphism between $E(Y_\chi)$ and $E(Y_{\chi_1}^{(1)}) \otimes_R E(Y_{\chi_2}^{(2)})$, suppose that $E(Y) = \sum_{n \in N}^{\oplus} R.A_n$ and $E(Y^{(i)}) = \sum_{n_i \in N^{(i)}}^{\oplus} R.A_{n_i}^{(i)}$, $i = 1,2$ (here $Y = RG[U]$ and $Y^{(i)} = RG^{(i)}[U^{(i)}]$), where A_n $(n \in N)$ and $A_{n_i}^{(i)}$ $(n_i \in N^{(i)})$ are defined as in §4.0. Let $n = (n_1, n_2) \in N$ $(= N^{(1)} \times N^{(2)})$, and consider the diagram

7.4.8 $$\begin{array}{ccc} Y & \xrightarrow{\alpha} & Y^{(1)} \otimes Y^{(2)} \\ {\scriptstyle A_n}\downarrow & & \downarrow {\scriptstyle A_{n_1}^{(1)} \otimes A_{n_2}^{(2)}} \\ Y & \xrightarrow[\alpha]{} & Y^{(1)} \otimes Y^{(2)} \end{array} ,$$

where $\alpha : Y \to Y^{(1)} \otimes Y^{(2)}$ is the RG-isomorphism which sends $[U]$ to $[U^{(1)}] \otimes [U^{(2)}]$ (in fact α is induced from $\Omega : RG \to RG^{(1)} \otimes RG^{(2)}$, defined in the proof of 7.4.3). We have

$$\begin{aligned}
\alpha A_n([U]) &= \alpha([UnU]) \\
&= \alpha([(U^{(1)},U^{(2)})(n_1,n_2)(U^{(1)},U^{(2)})]) \\
&= \alpha([(U^{(1)}n_1U^{(1)}\ ,\ U^{(2)}n_2U^{(2)})] \\
&= \alpha(([U^{(1)}n_1U^{(1)}]\ ,\ [U^{(2)}n_2U^{(2)}]) \\
&= [U^{(1)}n_1U^{(1)}] \otimes [U^{(2)}n_2U^{(2)}] \\
&= (A^{(1)}_{n_1} \otimes A^{(2)}_{n_2})\ \alpha\ ([U])\ .
\end{aligned}$$

Therefore $\alpha A_n = (A^{(1)}_{n_1} \otimes A^{(2)}_{n_2})\alpha$, and so the diagram 7.4.8 is commutative.

<u>Lemma 7.4.9</u> The map $A_{(n_1,n_2)} \to A^{(1)}_{n_1} \underset{R}{\otimes} A^{(2)}_{n_2}$ $(n_i \in N^{(i)}\ ,\ i = 1,2\ )$ defines an R-algebra isomorphism between $E(Y)$ and $E(Y^{(1)}) \underset{R}{\otimes} E(Y^{(2)})$.

<u>Proof</u> The R-algebras $E(Y)$ and $E(Y^{(1)}) \otimes E(Y^{(2)})$ are isomorphic, since $Y \cong Y^{(1)} \underset{R}{\otimes} Y^{(2)}$. The lemma follows from the commutativity of diagram 7.4.8. $\square$

We now return to the R-orders $E(Y_\chi)$ and $E(Y^{(1)}_{\chi_1}) \underset{R}{\otimes} E(Y^{(2)}_{\chi_2})$, where $\chi = (\chi_1,\chi_2) \in \tilde{H}$. We have

$$E(Y_\chi) \cong E_\chi = e_\chi\ E(Y)e_\chi = \sum_{w \in W_\chi}^{\oplus} R.e_\chi\ A_{(w)}\ ,$$

and, for $i = 1,2$,

$$E(Y_{\chi_i}) \cong E_{\chi_i} = e_{\chi_i}\, E(Y^{(i)})e_{\chi} = \sum^{\oplus}_{v\in W^{(i)}_{\chi_i}} R.e_{\chi_i}\, A^{(i)}_{(v)} \quad ,$$

where $e_{\chi_i} = |H^{(i)}|^{-1} \sum_{h\in H^{(i)}} \chi_i(h^{-1})A^{(i)}_h$ and W_χ , $W^{(i)}_{\chi_i}$ $(i = 1,2\ )$ are the subgroups of W , $W^{(i)}$, which stabilize χ , χ_i , respectively. If $w = (v_1,v_2) \in W$ then $w\chi = (v_1\chi_1,v_2\chi_2)$ and so $W_\chi = W^{(1)}_{\chi_1} \times W^{(1)}_{\chi_2}$.

<u>Proposition 7.4.10</u> The map $e_\chi A_{(w)} \to e_{\chi_1} A^{(1)}_{(v_1)} \otimes e_{\chi_2} A^{(2)}_{(v_2)}$ $(w = (v_1,v_2) \in W_\chi)$ defines an algebra isomorphism between E_χ and $E_{\chi_1} \underset{R}{\otimes} E_{\chi_2}$.

<u>Proof</u> We know that the R-orders $E_\chi = \sum^{\oplus}_{w\in W_\chi} R.e_\chi A_{(w)}$, and

$$E_{\chi_1} \underset{R}{\otimes} E_{\chi_2} = \sum^{\oplus}_{\substack{v_i\in W^{(i)}_{\chi_i}\\ i=1,2}} R.e_{\chi_1} A^{(1)}_{(v_1)} \underset{R}{\otimes} e_{\chi_2} A^{(2)}_{(v_2)}$$

are isomorphic as R-algebras.

The result follows since, for every $w = (v_1,v_2) \in W_\chi = W^{(1)}_{\chi_1} \times W^{(2)}_{\chi_2}$, the commutative diagram of 7.4.8 will induce a commutative diagram

7.4.11
$$\begin{tikzcd} Y_\chi \arrow[r, "\alpha"] \arrow[d, "e_\chi A_{(w)}"'] & Y^{(1)}_{\chi_1} \underset{R}{\otimes} Y^{(2)}_{\chi_2} \arrow[d, "e_{\chi_1} A^{(1)}_{(v_1)} \otimes e_{\chi_2} A^{(2)}_{(v_2)}"] \\ Y_\chi \arrow[r, "\alpha"] & Y^{(1)}_{\chi_1} \otimes Y^{(2)}_{\chi_2} \end{tikzcd}$$
. □

Now suppose that (χ,J) is an admissible $(G^{(1)} \times G^{(2)})$-pair, with $\chi = (\chi_1,\chi_2)$ and $\chi_i \in \tilde{H}^{(i)}$, $i = 1,2$. By 7.4.7(ii), we may assume that $J = J_1 \dot{\cup} J_2$, where $J_i \subseteq P(\chi_i)$, $i = 1,2$. Let $M(\chi,J)$, $M(\chi_i,J_i)$ be the simple $F(G^{(1)} \times G^{(2)})$ - , $FG^{(i)}$-module, which corresponds to the pairs (χ,J) and (χ_i,J_i) , $i = 1,2$, respectively (see §4.2). It is well-known (see [CRII], Thm. 10.33) that $M(\chi_1,J_1) \otimes_F M(\chi_2,J_2)$ is simple $F(G^{(1)} \times G^{(2)})$-module. Moreover, since F is a splitting field for $G^{(i)}$, $i = 1,2$, every simple $F(G^{(1)} \times G^{(2)})$-module is of the form $M(\chi_1,J_1) \otimes_F M(\chi_2,J_2)$ for some admissible $G^{(i)}$-pair (χ_i,J_i) , $i = 1,2$.

<u>Proposition 7.4.12</u> $M((\chi_1,\chi_2),J_1 \dot{\cup} J_2) \cong M(\chi_1,J_1) \otimes_F M(\chi_2,J_2)$ as $F(G^{(1)} \times G^{(2)})$-modules, for all admissible $(G^{(1)} \times G^{(2)})$-pair $((\chi_1,\chi_2), J_1 \dot{\cup} J_2)$.

<u>Proof</u> We have seen in (§4.2, Thm. 4.2.5) that $I_{U^{(i)}}(M(\chi_i,J_i))$ is a one-dimensional right $E(FY^{(i)})$-module affording the character $\psi(\chi_i,J_i)$ of $E(FY^{(i)})$, $i = 1,2$. That is $I_{U^{(i)}}(M(\chi_i,J_i)) = F.m_i$, where $m_i \in M(\chi_i,J_i)$, and

$$m_i \circ A^{(i)}_{h,F} = \chi_i(h) \qquad \text{for all } h \in H^{(i)} ,$$

and

7.4.13
$$m_i \circ A^{(i)}_{(w^{(i)}_j),F} = \begin{cases} 0 & \text{if } w^{(i)}_j \notin P(\chi_i)\backslash J_i \\ -m_i & \text{if } w^{(i)}_j \in P(\chi_i)\backslash J_i \end{cases} ,$$

for all $w^{(i)}_j \in \underline{R}^{(i)}$.

Since $M(\chi_1,J_1) \otimes_F M(\chi_2,J_2)$ is a simple $F(G^{(1)} \times G^{(2)})$-module, it follows from theorem 4.2.5 that $I_U(M(\chi_1,J_1) \underset{F}{\otimes} M(\chi_2,J_2))$

$$= I_{U^{(1)}}(M(\chi_1,J_1)) \underset{F}{\otimes} I_{U^{(2)}}(M(\chi_2,J_2))$$

$$= F.m_1 \otimes m_2$$

is a one-dimensional right $E(FY)$-module. To complete the proof it is enough to show that $F.m_1 \otimes m_2$ affords the character $\psi((\chi_1,\chi_2),J_1 \dot{\cup} J_2)$ of $E(Y)$. Let $n = (n_1,n_2) \in N = N^{(1)} \times N^{(2)}$. Recall (from 2.2.13) that the action of $A^{(i)}_{n_i,F} \in E(FY^{(i)})$ on $m_i \in I_{U^{(i)}}(M(\chi_i,J_i))$ is given by

$$m_i \circ A^{(i)}_{n_i,F} = a_{n_i} m_i \quad ,$$

where $a_{n_i} \in FG^{(i)}$ is such that $a_{n_i}[U^{(i)}] = A^{(i)}_{n_i,F}([U^{(i)}])$.

Since $\quad A^{(1)}_{n_1,F} \underset{F}{\otimes} A^{(2)}_{n_2,F}\ ([U^{(1)}] \otimes [U^{(2)}]) = a_{n_1}[U^{(1)}] \otimes a_{n_2}[U^{(2)}]$

7.4.14 $$= (a_{n_1} \otimes a_{n_2})([U^{(1)}] \otimes [U^{(2)}]) \ ,$$

it follows that

$$m_1 \otimes m_2 \circ A_{(n_1,n_2),F} = m_1 \otimes m_2 \circ A^{(1)}_{n_1,F} \otimes A^{(1)}_{n_2,F} \qquad \text{(see 7.4.8)}$$

$$= (a_{n_1} \otimes a_{n_2})(m_1 \otimes m_2)$$

$$= a_{n_1} m_1 \otimes a_{n_2} m_2$$

7.4.15 $$= m_1 \circ A^{(1)}_{n_1,F} \otimes m_2 \circ A^{(2)}_{n_2,F} \quad .$$

It is clear that, for all $h = (h_1,h_2) \in H$,

$$m_1 \otimes m_2 \circ A_{h,F} \ = \ (\chi_1,\chi_2)((h_1,h_2))\, m_1 \otimes m_2 \quad .$$

Let $w_i \in \underline{R} = \underline{R}^{(1)} \dot{\cup} \underline{R}^{(2)}$. If $w_i = w_i^{(1)} \in \underline{R}^{(1)}$, then, identifying $w_i^{(1)}$ with $(w_i^{(1)},1) \in W^{(1)} \times W^{(2)}$, we have

$$m_1 \otimes m_2 \circ A_{(w_i),F} = m_1 \otimes m_2 \circ A_{((w_i^{(1)}),1),F}$$

$$= m_1 \circ A_{(w_i^{(1)}),F} \otimes m_2 \circ A_{1,F} \qquad \text{by 7.4.15}$$

$$= m_1 \circ A_{(w_i^{(1)}),F} \otimes m_2$$

7.4.16
$$= \begin{cases} 0 & \text{if } w_i^{(1)} \notin P(\chi_1)\backslash J_1 \\ -m_1 \otimes m_2 & \text{if } w_i^{(1)} \in P(\chi_1)\backslash J_1 \end{cases}.$$

Similarly, if $w_i = w_i^{(2)} \in \underline{R}^{(2)}$, $1 \le i \le \ell_2$, then we will have

7.4.17
$$m_1 \otimes m_2 \circ A_{(w_i),F} = \begin{cases} 0 & \text{if } w_i^{(2)} \notin P(\chi_2)\backslash J_2 \\ -m_1 \otimes m_2 & \text{if } w_i^{(2)} \in P(\chi_2)\backslash J_2 \end{cases}.$$

Therefore, from 7.4.16 and 7.4.17, it follows that $F.m_1 \otimes m_2$ $(= I_U(M(\chi_1,J_1) \otimes M(\chi_2,J_2)))$ affords the character $\psi((\chi_1,\chi_2),J_1 \dot{\cup} J_2)$. Hence, by theorem 4.2.5, we conclude that

$$M(\chi_1,J_1) \underset{F}{\otimes} M(\chi_2,J_2) \cong M((\chi_1,\chi_2),J_1 \dot{\cup} J_2)$$

as $F(G^{(1)} \dot{\times} G^{(2)})$ -modules. □

Now let $\chi = (\chi_1,\chi_2)$ $\tilde{H}$ and let

7.4.18
$$Y_\chi = \sum_{J \subseteq \underline{P}(\chi)}^{\oplus} Y(\chi,J) ,$$

7.4.19
$$Y_{\chi_i}^{(i)} = \sum_{J \subseteq P(\chi_i)}^{\oplus} Y(\chi_i,J) , \quad i = 1,2 ,$$

be the decomposition of Y_χ and $Y^{(i)}_{\chi_i}$ into a direct sum of mutually non-isomorphic indecomposable RG- , $RG^{(i)}$-lattices, respectively, according to 6.0.10.

<u>Proposition 7.4.20</u> Let $\chi = (\chi_1,\chi_2) \in \tilde{H}$ and suppose that (χ,S) is an admissible $(G^{(1)} \times G^{(2)})$-pair. If $S = S_1 \cup S_2$, then

$$Y(\chi,S) \cong Y(\chi_1,S_1) \otimes_R Y(\chi_2,S_2) .$$

To prove 7.4.20, we need the following

<u>Lemma 7.4.21</u> Let $G = (G,B,N,\underline{R},U)$ be any finite group with a split BN-pair of characteristic p . Let (χ,S) be an admissible G-pair, and $Y(\chi,S)$ be the indecomposable RG-summand of $Y = RG[U]$ which corresponds to (χ,S) . Then

$$E(FY(\chi,S))/\underline{r}(E(FY(\chi,S)) \cong F .$$

<u>Proof</u> Let $e = e(\chi,S)$ be the primitive idempotent of $E(FY)$ which corresponds to the indecomposable FG-summand $FY(\chi,S)$ of FY . Then we have $E(FY(\chi,S)) \cong eE(FY)e$ (see appendix, lemma 1), and hence $\underline{r}(E(FY(\chi,S)) \cong e\underline{r}(E(FY))e$ (see [CRII], Prop. 5.13). The right $E(FY)$-module $eE(FY)$ is a projective cover of the one-dimensional right $E(FY)$-module $\mathcal{S}(\chi,S)$ which affords the character $\psi(\chi,S)$ (see 4.2.4). Hence $eE(FY) / e\underline{r}(E(FY))$ $(\cong \mathcal{S}(\chi,S))$ is also one-dimensional.

The map

$$x + e\underline{r}(E(FY)) \to xe + e\underline{r}(E(FY))e \quad ,$$

$x \in eE(FY)$, is clearly an F-isomorphism between $eE(FY) / e\underline{r}(E(FY))$

and $eE(FY)e \,/\, e\underline{r}(E(FY))e$. Therefore

$$E(FY(\chi,S))/\underline{r}(E(FY(\chi,S)) \quad (\cong eE(FY)e \,/\, e\underline{r}(E(FY))e)$$

is one-dimensional, hence

$$E(FY(\chi,S)) \,/\, \underline{r}(E(FY(\chi,S)) \cong F \ . \qquad \square$$

Proof of 7.4.20 It is clear that $Y(\chi_1,S_1) \otimes_R Y(\chi_2,S_2)$ is an $R(G^{(1)} \times G^{(2)})$-summand of $Y^{(1)} \otimes Y^{(2)}$ $(\cong Y)$.

(i) We first show that $Y(\chi_1,S_1) \otimes Y(\chi_2,S_2)$ is indecomposable $R(G^{(1)} \times G^{(2)})$-lattice by showing that $FY(\chi_1,S_1) \otimes_F FY(\chi_2,S_2)$ $(\cong F \otimes_R (Y(\chi_1,S_1) \otimes_R Y(\chi_2,S_2))$ is indecomposable. Consider the endomorphism algebra $E(FY(\chi_1,S_1) \otimes_F FY(\chi_2,S_2))$. We have

$$E(FY(\chi_1,S_1) \otimes_F FY(\chi_2,S_2)) \cong E(FY(\chi_1,S_1) \otimes_F E(FY(\chi_2,S_2))$$

(see [CRII], 10.37). The ideal

$$M = \underline{r}(E(FY(\chi_1,S_1)) \otimes_F E(FY(\chi_2,S_2)) + E(FY(\chi_1,S_1) \otimes_F \underline{r}(E(FY(\chi_2,S_2))$$

is a nilpotent ideal of $E(FY(\chi_1,S_1)) \otimes_F E(FY(\chi_2,S_2)$, hence $M \le \underline{r}(E(FY(\chi_1,S_1)) \otimes_F E(FY(\chi_2,S_2))$. On the other hand

$$\begin{aligned} &E(FY(\chi_1,S_1) \otimes E(FY(\chi_2,S_2) \,/\, M \\ &\qquad \cong E(FY(\chi_1,S_1))/\underline{r}E(FY(\chi_1,S_1) \otimes E(FY(\chi_2,S_2))/\underline{r}(E(FY(\chi_2,S_2)) \\ &\qquad\qquad \text{(see [CRII], proof of 10.38)} \\ &\qquad \cong F \otimes F \cong F \ , \quad \text{by 7.4.21} \end{aligned}$$

is semisimple. Therefore, M contains $\underline{r}(E(FY(\chi_1,S_1) \otimes_F E(FY(\chi_2,S_2))$, hence $M = \underline{r}(E(FY(\chi_1,S_1)) \otimes E(FY(\chi_2,S_2))$. Since $E(FY(\chi_1,S_1)) \otimes_F E(FY(\chi_2,S_2)) / M \cong F$, it follows that $E(FY(\chi_1,S_1) \otimes_F FY(\chi_2,S_2))$ $(\cong E(FY(\chi_1,S_1) \otimes FY(\chi_2,S_2))$ is local algebra, and so $FY(\chi_1,S_1) \otimes_F FY(\chi_2,S_2)$ is indecomposable $F(G^{(1)} \times G^{(2)})$-module (see [CRII], Cor. 6.4). Consequently $Y(\chi_1,S_1) \otimes Y(\chi_2,S_2)$ is indecomposable $R(G^{(1)} \times G^{(2)})$-lattice.

(ii) We have
$$Y_\chi \cong \sum^{\oplus}_{J\subseteq P(\chi)} Y(\chi,J)$$
$$\cong Y^{(1)}_{\chi_1} \otimes Y^{(2)}_{\chi_2} \quad , \text{ by } 7.4.3$$
$$\cong \sum^{\oplus}_{\substack{J_1\subseteq P(\chi_1)\\ J_2\subseteq P(\chi_2)}} Y(\chi_1,J_1) \otimes Y(\chi_2,J_2) \ .$$

Since $Y(\chi_1,S_1) \otimes Y(\chi_2,S_2)$ is an indecomposable $R(G^{(1)} \times G^{(2)})$-lattice, it follows, using Krull-Schmidt theorem, that $Y(\chi_1,S_1) \otimes Y(\chi_2,S_2) \cong Y(\chi,J)$ for a unique $J \subseteq P(\chi)$. But since the head of $FY(\chi_1,S_1) \otimes_F FY(\chi_2,S_2)$ is isomorphic to $M(\chi_1,S_1) \otimes_F M(\chi_2,S_2)$ $(= M(\chi, S_1 \dot{\cup} S_2)$, by 7.4.12), we must have $J = S_1 \dot{\cup} S_2$, and so

$$Y(\chi_1,S_1) \otimes Y(\chi_2,S_2) \cong Y(\chi, S_1 \dot{\cup} S_2) \ .$$

This completes the proof of 7. 4.20. □

Let $\chi = (\chi_1,\chi_2) \in \hat{H}$. For $i = 1,2$, let $\{X^{(i)}_j , j \in I_i\}$ be the full set of simple $KG^{(i)}$-components of $KY^{(i)}_{\chi_i}$. Then $\{X^{(1)}_i \otimes X^{(2)}_j \mid (i,j) \in I_1 \times I_2\}$ is a full set of simple $K(G^{(1)} \times G^{(2)})$-components of $KY^{(1)}_{\chi_1} \otimes KY^{(2)}_{\chi_2}$. Since $KY_\chi \cong KY^{(1)}_{\chi_1} \otimes_K KY^{(2)}_{\chi_2}$, the simple $K(G^{(1)} \times G^{(2)})$-

components of KY_χ are also indexed by the set $I_1 \times I_2$. Let $\{X_{i,j}, (i,j) \in I_1 \times I_2\}$ be the full set of simple $K(G^{(1)} \times G^{(2)})$-components of KY_χ and assume that $X_{i,j}$ $((i,j) \in I_1 \times I_2)$ is taken to $X_i^{(1)} \underset{K}{\otimes} X_j^{(2)}$ under the isomorphism $KY \cong KY_{\chi_1}^{(1)} \underset{K}{\otimes} KY_{\chi_2}^{(2)}$.

For $(i,j) \in I_1 \times I_2$ and $J \subseteq P(\chi)$, let $d_{(i,j),J}$ denote the $((i,j),J)$-entry of the decomposition matrix of the system $(KE_\chi, E_\chi, FE_\chi)$. Similarly, for $j \in I_i$ $(i = 1,2)$ and $S \subseteq P(\chi_i)$, let $d_{i,S}$ denote the (i,S)-entry of the decomposition matrix of the system $(KE_{\chi_i}, E_{\chi_i}, FE_{\chi_i})$.

Proposition 7.4.22 Let $\chi = (\chi_1, \chi_2) \in \tilde{H}$, I_1 and I_2 be as above. Let $S = S_1 \dot{\cup} S_2$ be a subset of $P(\chi)$. Then, for all $(i,j) \in I_1 \times I_2$,

$$d_{(i,j),S} = d_{i,S_1} \cdot d_{i,S_2}$$

Proof From theorem 7.2.3, we have

$$\begin{aligned} d_{(i,j),S} &= d^*_{(i,j),S} \\ &= [X_{i,j} \mid KY(\chi,S)] \\ &= [X_i^{(1)} \otimes X_j^{(2)} \mid KY(\chi_1,S_1) \otimes KY(\chi_2,S_2)], \quad \text{by 7.4.20.} \end{aligned}$$

For $i = 1,2$, let η_i be the K-character of $G^{(i)}$ afforded by $KY(\chi_i,S_i)$. Let $\theta_i^{(1)}$, $\theta_j^{(2)}$ be the irreducible K-character of $G^{(1)}$, $G^{(2)}$, which is afforded by $X_i^{(1)}$, $X_j^{(2)}$, respectively. Then the $K(G^{(1)} \times G^{(2)})$-module $KY(\chi_1,S_1) \otimes KY(\chi_2,S_2)$ affords the K-character $\eta_1\eta_2$ which is given by

$$\eta_1\eta_2\,((g_1,g_2)) := \eta_1(g_1)\eta_2(g_2),$$

for all $(g_1,g_2) \in G^{(1)} \times G^{(2)}$. Similarly, the simple $K(G^{(1)} \times G^{(2)})$-module $X_i^{(1)} \otimes X_j^{(2)}$ affords the character $\theta_i^{(1)}\theta_j^{(2)}$. It is well-known (see [CRII], Prop. 9.23(ii)) that

$$[X_i^{(1)} \otimes X_j^{(2)} \mid KY(\chi_1,S_1) \otimes KY(\chi_2,S_2)] = \langle\theta_i^{(1)}\theta_j^{(2)}, \eta_1\eta_2\rangle_{G^{(1)} \times G^{(2)}} .$$

But $\langle\theta_i^{(1)}\theta_j^{(2)}, \eta_1\eta_2\rangle_{G^{(1)} \times G^{(2)}}$

$$= \frac{1}{|G^{(1)}\times G^{(2)}|} \sum_{(g_1,g_2)\in G^{(1)}\times G^{(2)}} \theta_i^{(1)}\theta_j^{(2)}((g_1,g_2)).\eta_1\eta_2((g_1^{-1},g_2^{-1}))$$

$$= |G^{(1)}|^{-1}|G^{(2)}|^{-1} \sum_{(g_1,g_2)\in G^{(1)}\times G^{(2)}} \theta_i^{(1)}(g_1)\theta_j^{(2)}(g_2)\eta_1(g_1^{-1})\eta_2(g_2^{-1})$$

$$= \langle\theta_i^{(1)}, \eta_1\rangle_{G^{(1)}} \cdot \langle\theta_j^{(2)}, \eta_2\rangle_{G^{(2)}}$$

$$= [X_i^{(1)} \mid KY(\chi_1,S_1)].[X_j^{(2)} \mid KY(\chi_2,S_2)] .$$

Therefore

$$d_{(i,j),S} = [X_i^{(1)} \otimes X_j^{(2)} \mid KY(\chi_1,S_1) \otimes KY(\chi_2,S_2)]$$

$$= [X_i^{(1)} \mid KY(\chi_1,S_1)]\,[X_j^{(2)} \mid KY(\chi_2,S_2)]$$

$$= d^*_{i,S_1} \cdot d^*_{j,S_2}$$

$$= d_{i,S_1} \cdot d_{i,S_2} \qquad \text{by theorem 7.2.3.}$$

This completes the proof of 7.4.22. $\square$

§7.5 The Case Of The General Linear Group

Let $G = GL(n,q)$, the general linear group over a finite field $\mathbb{F}_q$ $(= GF(q))$, where q is a power of a prime $p > 0$. Let $(G,B,N,\underline{R},U)$ be the split BN-pair in G defined in 1.2.5.

In this section we apply the results of Chapters 6 and 7 to the case $G = GL(n,q)$. In particular we will see that the problem of calculating the decomposition numbers $d_{i,(\chi,J)}$ (see 7.2.3) can be reduced to the case when $\chi = 1$.

Let (K,R,F) be a p-modular system such that K is a splitting field for G and all its subgroups. Let $\Gamma : (\mathbf{F}_q^\times)_{q-1} \to (K^\times)_{q-1}$ be as in 4.1.14. Let s be a multiplicative generator of $\mathbf{F}_q^\times$, and for $i \in \{1,2,\ldots,n\}$, let

$$s_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & s & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}_i$$

. It is clear that

$H = B \cap N \cong \langle s_1\rangle \times \langle s_2\rangle \times\ldots\times \langle s_n\rangle$, where $\langle s_i\rangle$ is a cyclic group of order $q-1$, for all $1 \le i \le n$. It follows from the representation theory of abelian groups (see [CRI], p.37) that the set $\bar{H} = \mathrm{Hom}(H,F^\times)$ of multiplicative characters of H is indexed by the set of all n-tuples $(a_1,\ldots,a_n)$, where $0 \le a_i < q-1$, for all $i \in \{1,2,\ldots,n\}$. In fact

$$\bar{H} = \{\chi(a_1,\ldots,a_n) \mid 0 \le a_i < q-1\ , \text{ for all } i \in \{1,\ldots,n\}\}\ ,$$

where $\chi(a_1,\ldots,a_n) \in \bar{H}$ is given by:

7.5.1 $$\chi(a_1,\ldots,a_n) : \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots \; x_n \end{pmatrix} \mapsto \prod_{i=1}^{n} x_i^{a_i}\ ,$$

for all $\begin{pmatrix} x_1 & & & \\ & x_1 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \in H$. It is sometimes convenient to identify $a_1,\ldots,a_n$ with their residue classes $\mathrm{mod}(q-1)$.

For every n-tuple $(a_1,\ldots,a_n)$, $0 \leq a_i < q-1$ let $\chi^{o}(a_1,\ldots,a_n)$ be the element of $\tilde{H} = \mathrm{Hom}(H,K^{\times})$ given by

7.5.2 $$\chi^{o}(a_1,\ldots,a_n) : \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \to \prod_{i=1}^{n} \Gamma(x_i)^{a_i} ,$$

for all $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in H$. $\chi^{o}(a_1,\ldots,a_n)$ is the Brauer lift for $\chi(a_1,\ldots,a_n)$ (i.e. $\overline{\chi^{o}(a_1,\ldots,a_n)} = \chi(a_1,\ldots,a_n)$) , and the map

7.5.3 $$\chi(a_1,\ldots,a_n) \to \chi^{o}(a_1,\ldots,a_n)$$

gives a bijection between $\bar{H}$ and $\tilde{H}$.

The Weyl group

$$W \cong S_n = \langle w_1 = (12),\ w_2 = (23)\ ,\ldots,\ w_{n-1} = (n\text{-}1\ n)\rangle$$
$$= \langle \underline{R} \rangle \quad .$$

W acts on $\bar{H}$ as follows: if $\chi(a_1,\ldots,a_n) \in \bar{H}$, and $w \in W$, then

7.5.4 $$w\chi(a_1,\ldots,a_n) := \chi(a_{w^{-1}(1)}\ ,\ldots,\ a_{w^{-1}(n)}) .$$

Therefore

$$W_{\chi(a_1,\ldots,a_n)} = \{w \in W \mid a_{w(i)} \equiv a_i \pmod{q-1}\ ,\ \text{for all}\ \ i \in \{1,2,\ldots,n\}\} .$$

For every $i \in \{1,\ldots,n-1\}$, we can take

$$(w_i) = \begin{array}{c|ccc|c}
 & & i & i+1 & \\
 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & & \\
i & & 0 & -1 & \\
i+1 & & 1 & 0 & \\
 & & & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}
\end{array}$$

to be the coset representative in N of w_i, i.e. $(w_i)H = w_i \in W$, for all $w_i \in \underline{R}$. For every $i \in \{1,\ldots,n-1\}$, we have

$$U_i = \left\{ \begin{pmatrix} 1 & & & \\ & \ddots\ 1 & \lambda & \\ & & 1 & \\ & & & \ddots\ 1 \end{pmatrix} \begin{matrix} i \\ i+1 \end{matrix} , \lambda \in \mathbb{F}_q \right\}.$$

Hence

$$\langle U_i, U_{-i} \rangle = \begin{array}{c} \\ \left(\begin{array}{c|c|c} 1 \ddots 1 & & \\ \hline & SL(2,q) & \\ \hline & & 1 \ddots 1 \end{array}\right) \end{array} \begin{matrix} i \\ i+1 \end{matrix} , \text{ and so}$$

$$H_i = H \cap \langle U_i, U_{-i} \rangle$$

$$= \left\{ \begin{matrix} \\ i \\ i+1 \\ \\ \end{matrix} \begin{pmatrix} 1 \ddots 1 & & & \\ & x & & \\ & & x^{-1} & \\ & & & 1 \ddots 1 \end{pmatrix} , x \in \mathbb{F}_q^{\times} \right\}.$$

Therefore, if $\chi = \chi(a_1,\ldots,a_n) \in \bar{H}$, then

$P(\chi) = \{w_i \in \underline{R} \mid a_i \equiv a_{i+1} \pmod{q-1}\}$.

The structural equations for $GL(n,q)$

For every $w_i \in \underline{R}$ and all $x \in \mathbb{F}_q^{\times}$, we have

$$7.5.5 \quad (w_i)^{-1}\begin{pmatrix}1 & & \\ & \ddots & x \\ & & 1\end{pmatrix}(w_i) = \begin{pmatrix}1 & & \\ & \ddots & -x^{-1} \\ & & 1\end{pmatrix}\begin{pmatrix}1 \ddots & & & \\ & x & & \\ & & x^{-1} & \\ & & & \ddots 1\end{pmatrix}(w_i)\begin{pmatrix}1 & & \\ & \ddots & -x^{-1} \\ & & 1\end{pmatrix} .$$

We denote $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & x & & \\ & & & & x^{-1} & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}$ by $h_i(x)$.

Let $\chi = \chi(a_1,\dots,a_n) \in \bar{H}$. Define a relation $\sim$ on the set $\underline{n} = \{1,2,\dots,n\}$ as follows: if $i,j \in \underline{n}$, then $i \sim j \iff a_i \equiv a_j$ (mod q-1) . $\sim$ is clearly an equivalence relation. Let $A_1 ,\dots, A_d$ be the set of equivalence $\sim$-classes of $\underline{n}$. Then it is clear that

$$W_{\chi(a_1,\dots,a_n)} = S(A_1) \times\dots\times S(A_d) ,$$

where, for all $i \in \{1,2,\dots,d\}$,

$$S(A_i) := \{w \in W \mid w(j) = j , \text{ for all } j \in \underline{n}\backslash A_i\} .$$

Since $w\chi(a_1,\dots,a_n) = \chi(a_{w^{-1}(1)},\dots, a_{w^{-1}(n)})$, for all $w \in W$, by changing to another element of the W-orbit of $\chi(a_1,\dots,a_n)$, we can arrange so that each A_j , $j \in \underline{d}$, is an interval in $\{1,2,\dots,n\}$, i.e. $A_j = \{i, i+1, \dots, m-1, m\}$ for some $1 \le i < m \le n$. In this case we have

$$W_{\chi(a_1,\dots,a_n)} = \langle P(\chi(a_1,\dots,a_n))\rangle .$$

The following lemma summarizes the above argument.

<u>Lemma 7.5.6</u> For every $\chi \in \bar{H}$, there is an element $\lambda \in (\chi)$ with

$$W_\lambda = W_{P(\lambda)} ,$$

where (χ) is the W-orbit of χ in $\bar{H}$. $\square$

Remark 7.5.7 If $\chi(a_1,\dots,a_n) \in \bar{H}$ and $\chi^o(a_1,\dots,a_n) \in \tilde{H}$ is its Brauer lift, then, for $w \in W$, we have $w\chi^o(a_1,\dots,a_n) = \chi^o(a_{w^{-1}(1)},\dots,a_{w^{-1}(n)})$.

Hence
$$\begin{aligned}\overline{w\chi^o(a_1,\dots,a_n)} &= \overline{\chi^o(a_{w^{-1}(1)},\dots,a_{w^{-1}(n)})} \\ &= \chi(a_{w^{-1}(1)},\dots,a_{w^{-1}(1)}) \\ &= w\chi(a_1,\dots,a_n) \ .\end{aligned}$$

Therefore the argument of lemma 7.5.6 can be applied also to the W-orbit of $\tilde{H} = \mathrm{Hom}(H,K^\times)$.

It follows from lemma 7.5.6 that, since $KY_\chi \cong KY_{w\chi}$ as KG-modules, for all $\chi \in \tilde{H}$ and all $w \in W$, we may (and will) assume that

$$\chi = \chi^o(a_1,\dots,a_1,\ a_2,\dots,a_2,\dots,a_d,\dots\dots a_d) \ ,$$

where, for all $1 \le i \le d$, $\#\ a_i$'s $= |A_i| = \ell_i$, say, and $a_i \not\equiv a_j \pmod{q-1}$ for all $1 \le i \ne j \le d$. For every $i \in \{1,2,\dots,d\}$, let $\hat{\ell}_i = \ell_1 +\dots+ \ell_i$. Then it is clear that $P(\chi) = \underline{R}\backslash\{w_{\hat{\ell}_1}, w_{\hat{\ell}_2},\dots,w_{\hat{\ell}_{d-1}}\}$, and that $W_\chi = W_{P(\chi)}$. By 7.2.11, the decomposition matrix D_χ of the system $(KE_\chi, E_\chi, FE_\chi)$ is identical with the decomposition matrix $D_{\chi,J}$ of the system $(KE_{\chi,J}, E_{\chi,J}, FE_{\chi,J})$, where $J = P(\chi)$. (Recall $E_{\chi,J} \cong \mathrm{End}_{RG_J}(Y_{\chi,J}))$.

The parabolic subgroup $G_{P(\chi)}$ is of the form

$$\left(\begin{array}{c|cc} GL(\ell_1,q) & * & * \\ \hline 0 & GL(\ell_2,q) & \\ & \ddots & * \\ 0 & 0 & GL(\ell_d,q) \end{array}\right)$$

with Levi subgroup

7.5.8 $$L_{P(\chi)} = \begin{pmatrix} GL(\ell_1,q) & & & \\ & GL(\ell_2,q) & & \\ & & \ddots & \\ & & & GL(\ell_d,q) \end{pmatrix} \cong G^{(1)} \times \dots \times G^{(d)} ,$$

where $G^{(i)} = GL(\ell_i,q)$. For $i \in \{1,2,\dots,d\}$, let $(G^{(i)},B^{(i)},N^{(i)},\underline{R}^{(i)},U^{(i)})$ be the usual split BN-pair defined in $G^{(i)}$, and let $H^{(i)} = B^{(i)} \cap N^{(i)}$. By 7.4.1, $(\prod_{i=1}^{d} G^{(i)} , \prod_{i=1}^{d} B^{(i)} , \prod_{i=1}^{d} N^{(i)} , \dot{\cup}\ \underline{R}^{(i)} , \prod_{i=1}^{d} U^{(i)})$ forms a split BN-pair in $\prod_{i=1}^{d} G^{(i)}$ with $\prod_{i=1}^{d} N^{(i)} \cap \prod_{i=1}^{d} B^{(i)} = \prod_{i=1}^{d} H^{(i)}$ $(\cong H)$. The Borel subgroup $B_{P(\chi)}$ of $L_{P(\chi)}$ has the form

$$\begin{pmatrix} B^{(1)} & 0 & 0 \\ 0 & B^{(2)} & \\ & & \ddots & 0 \\ 0 & & 0 & B^{(d)} \end{pmatrix} \cong B^{(1)} \times \dots \times B^{(d)} .$$

From 7.2.12 we have $Y_{\chi,P(\chi)} \cong (Y_{\chi}^{P(\chi)})_{G_{P(\chi)}}$ as $RG_{P(\chi)}$-lattice and by 7.2.20, the decomposition matrix $D_{\chi,P(\chi)}$ of the system $(KE_{\chi,P(\chi)}, E_{\chi,P(\chi)}, FE_{\chi,P(\chi)})$ is identical to the decomposition matrix D_{χ}^{J} of the system $(KE_{\chi}^{J}, E_{\chi}^{J}, FE_{\chi}^{J})$ (Recall $E_{\chi}^{J} \cong \mathrm{End}_{RL_J}(Y_{\chi}^{J})$) , where $J = P(\chi)$.

Consider the RL_J-lattice $Y_{\chi}^{J} = RL_J[U]\beta_{\chi}$. For each $i \in \{1,2,\dots,d\}$, let $\chi_i = \chi(\underbrace{a_i,a_i,\dots,a_i}_{\ell_i}) \in \tilde{H}^{(i)}$, and define $\chi' = (\chi_1,\chi_2,\dots,\chi_d)$.

It is clear that χ' is a multiplicative character of $\prod_{i=1}^{d} H^{(i)}$ and that

7.5.9 $$Y_\chi^{P(\chi)} \cong R(\prod_i G^{(i)})\,[U]\beta_\chi$$
$$\cong \bigotimes_{i=1}^{d} RG^{(i)}\,[U^{(i)}]\beta_{\chi_i} \qquad \text{by 7.4.3} .$$

It follows from 7.4.22 that the decomposition matrix $D_{\chi,P(\chi)}$ can be determined from the decomposition matrices D_{χ_i} $(1 \le i \le n)$ of the system $(E(KY_{\chi_i}^{(i)}, E(Y_{\chi_i}^{(i)}), E(FY_{\chi_i}^{(i)}))$, where $Y_{\chi_i}^{(i)} = RG^{(i)}[U^{(i)}]\beta_{\chi_i}$. Since $\chi_i = \chi(a_i,\ldots,a_i)$, $P(\chi_i) = \underline{R}^{(i)}$ for each $1 \le i \le d$.

Therefore we may assume that $\chi \in \tilde{H}$ with $P(\chi) = \underline{R}$. Since $Y_\chi \cong Y(\chi,\underline{R}) \otimes_R Y_1$, where $Y_1 = RG[B]$ (see 6.0.20), we may take $\chi = 1$.

Consider the decomposition matrix D_1 of the system (KE_1, E_1, FE_1) . KE_1 is anti-isomorphic to the Hecke algebra $H_K(q)$ of the BN-pair $(G,B,N,\underline{R},U)$. Since $H_K(q) \cong KS_n$, it follows that the rows of D_1 are indexed by the set of irreducible K-characters of S_n . It is well-known (see [GJ]) that the set of irreducible K-characters of S_n are indexed by the set of partitions

$$\mathcal{P} = \{\lambda = (\lambda_1 \ge \lambda_2 \ge \ldots \quad) \,/\, \lambda_i \in \mathbb{Z}_{\ge 0} \quad \text{for all } i$$
$$\text{and } \sum_i \lambda_i = n \} .$$

Let $\{\xi_\lambda, \lambda \in \mathcal{P}\}$ be the full set of irreducible characters of $W = S_n$. The set $\mathcal{P}$ can be used also to index the set of irreducible K-characters of G which appear in 1_B^G . Let $\{\theta_\lambda, \lambda \in \mathcal{P}\}$ be the full set of irreducible K-characters of G which appear in 1_B^G . Since $P(1) = \underline{R}$, the columns of D_1 are indexed by the subsets of $\underline{R}$. If $\lambda \in \mathcal{P}$ and $J \subseteq \underline{R}$, let $d_{\lambda,J}$ denote the (λ,J)-entry of D_1 . It follows from 7.2.31 that, for all $\lambda \in \mathcal{P}$ and all $J \subseteq \underline{R}$, 7.5.10 $\quad d_{\lambda,J}\ (= \langle\theta_\lambda,\eta_J\rangle) = \langle\xi_\lambda,\omega_J\rangle$,

where $\eta_J \in Ch_K G$ and $\omega_J \in Ch_K W$ are given by

$$\eta_J = \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|}\, 1_{G_S}^{G} \quad , \text{ and}$$

$$\omega_J = \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|}\, 1_{W_S}^{W}$$

For each $S \subseteq \underline{R}$ we associate a graph with nodes labelled $1,2,\ldots,n$ in which nodes i , $i+1$ are joined if $w_i \in S$. $S \subseteq \underline{R}$ is called <u>connected</u> if the graph associated with S is connected in the usual sense, that is if the set $\{i \mid w_i \in S\}$ is an interval in $\{1,2,\ldots,\ell\}$. In this way we can write any subset S of $\underline{R}$ as a union of connected subsets by looking at its corresponding graph.

<u>Example 7.5.11</u> Let $n = 5$ and $S = \{w_1,w_3,w_4\}$. Then the graph of S is

$$\overset{1}{\times} \text{———} \overset{2}{\times} \qquad \overset{3}{\times} \text{———} \overset{4}{\times} \text{———} \overset{5}{\times} \quad .$$

We also have $S = S_1 \cup S_2$ where $S_1 = \{w_1\}$ and $S_2 = \{w_3,w_4\}$. Both S_1 and S_2 are connected subsets of S in the above sense.

<u>Definition 7.5.12</u> For each $S \subseteq \underline{R}$, we associate a partition $\lambda(S)$ of n as follows: Write $S = S_1 \,\dot\cup\, S_2 \,\dot\cup \ldots \dot\cup\, S_r$ as a disjoint union of connected subsets of S , and assume that $|S_1| \geq |S_2| \geq \ldots \geq |S_r|$. Define $\lambda(S)$ to be the partition $(|S_1|+1\ ,\ |S_2|+1,\ldots,\ |S_r|+1,\ 1^{n_S})$, where $n_S = n - \sum_{i=1}^{r} |S_i| + 1$.

For example in 7.5.11 the partition $\lambda(S)$ of $n = 5$, associated with $S = \{w_1, w_3, w_4\}$, is $(3,2)$.

<u>Definition 7.5.13</u> (See [GJ], p.8): Let $\lambda \in P$.

(i) The <u>Young diagram</u> $[\lambda]$ is $\{(i,j) | i,j \in \mathbb{Z}, 1 \le i , 1 \le j \le \lambda_i\}$. If $(i,j) \in [\lambda]$, then (i,j) is called a <u>node</u> of $[\lambda]$.

(ii) A <u>λ-tableau</u> is one of the $n!$ arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integers $1,2,\ldots, n$ allowing no repeats.

For example if $n = g$ and $\lambda = (4,2^2,1)$, then $[\lambda] = \begin{matrix} \times & \times & \times & \times \\ \times & \times & & \\ \times & \times & & \\ \times & & & \end{matrix}$

and $\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 4 & & \\ 7 & 8 & & \\ 9 & & & \end{matrix}$ is a λ-tableau.

For each partition $\lambda = (\lambda_1,\lambda_2,\ldots)$ of n , let

$$W(\lambda) = W\{1,\ldots,\lambda_1\} \times W\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\} \times \ldots ,$$

where $W\{1,\ldots,\lambda_1\}$ denotes the permutation group on the letters $1,\ldots,\lambda_1,\ldots$ etc.. $W(\lambda)$ is a subgroup of W called <u>Young subgroup</u> of W of type λ . For example if $n = 7$ and $\lambda = (2^2,3)$ then

$$W(\lambda) = W\{1,2\} \times W\{3,4\} \times W\{5,6,7\} .$$

It is clear that if $S \subseteq \underline{R}$, then $W(\lambda(S)) \cong W_S$ $(= \langle S \rangle)$. Therefore if we let ϕ^λ $(\lambda \in P)$ denote the induced character $1_{W(\lambda)}^W$, then $\phi^{\lambda(S)} = 1_{W_S}^W$. It follows that for every $J \subseteq \underline{R}$,

7.5.14 $$\omega_J = \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\setminus J|} \phi^{\lambda(S)} .$$

Writing $\phi^{\lambda(S)}$ $(S \subseteq \underline{R})$ as a combination of irreducible characters can be achieved using Young's rule. For that we need to use tableaux with repeated entries as the following definition suggests.

Definition 7.5.15 (See [GJ], p.44):

(i) A tableau T is of type μ, for some $\mu \in P$, if for each i, the number i occurs μ_i times in T.

For example $\begin{matrix} 2 & 2 & 1 & 1 \\ 3 & 1 & & \\ 4 & & & \end{matrix}$ is $(4,2,1)$-tableau of type $(3,2,1^2)$.

(ii) A tableau T is semistandard if the numbers are non-decreasing along the rows of T and strictly increasing down columns of T.

The set of partitions of n is partially ordered by the following order: If $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ are partitions of n, then we write $\mu \trianglelefteq \lambda$ provided that for all j

$$\sum_{i=1}^{j} \mu_i \le \sum_{i=1}^{j} \lambda_i .$$

If $\mu \le \lambda$ and $\lambda \ne \mu$ we write $\mu \triangleleft \lambda$.

The proof of the following proposition can be found in James [GJ].

Proposition 7.5.16 If $\lambda \in P$, then:

(i) (Young's Rule): $\phi^\lambda = \sum_{\mu \in P} m_{\lambda,\mu} \xi_\mu$, where for each $\mu \in P$, $m_{\lambda,\mu}$ is the number of μ-tableaux of type λ,

(ii) $\langle \xi_\lambda, \phi^\lambda \rangle_W = 1$.

(iii) If $\lambda \ne \mu$ then

$$\langle \xi_\mu, \phi^\lambda \rangle_W \ne 0 \Rightarrow \lambda \triangleleft \mu . \qquad \square$$

Lemma 7.5.17 If $J_1, J_2 \subseteq \underline{R}$ with $J_1 \subseteq J_2$ then $\lambda(J_1) \trianglelefteq \lambda(J_2)$. $\square$

Proposition 7.5.18 For every $J \subseteq \underline{R}$, we have

$$\langle \xi_{\lambda(J)}, \omega_J \rangle_W = 1 .$$

Proof We have from 7.5.14

$$\omega_J = \sum_{J\subseteq S\subseteq \underline{R}} (-1)^{|S\backslash J|} \phi^{\lambda(S)}$$

$$= \phi^{\lambda(J)} + \sum_{J\subset S\subseteq \underline{R}} (-1)^{|S\backslash J|} \phi^{\lambda(S)} .$$

Hence

$$<\xi_{\lambda(J)},\omega_J>_W = <\xi_{\lambda(J)},\phi^{\lambda(J)}>_W + \sum_{J\subset S\subseteq \underline{R}} (-1)^{|S\backslash J|} <\xi_{\lambda(J)},\phi^{\lambda(S)}>_W .$$

By 7.5.17, $J \subset S \Rightarrow \lambda(J) \lhd \lambda(S)$, and by 7.5.16(iii) we have

$$J \subset S \Rightarrow <\xi_{\lambda(J)},\phi^{\lambda(S)}>_W = 0 .$$

Therefore, by 7.5.16 we have

$$<\xi_{\lambda(J)},\omega_J>_W = <\xi_{\lambda(J)},\phi^{\lambda(J)}>_W = 1 . \qquad \square$$

For a subset J of $\underline{R}$, let $\bar{J} = \underline{R}\backslash J$. Let $J \subseteq \underline{R}$ and consider the induced RG-lattice $Y_{\bar{J}}(1,\phi)^G$. By 6.0.10 we have

7.5.19 $$Y_{\bar{J}}(1,\phi)^G \cong \sum_{\substack{L\subseteq \underline{R} \\ L\cap\bar{J}=\phi}} Y(1,L) = \sum_{L\subseteq J} Y(1,L) .$$

We have seen in 6.0.20 that the $KG_{\bar{J}}$-module $KY_{\bar{J}}(1,\phi)$ affords the Steinberg character $St_{G_{\bar{J}}}$. Therefore taking characters, in 7.5.19, we have

7.5.20 $$St_{G_{\bar{J}}}^G = \sum_{L\subseteq J} \eta_L .$$

On the other hand, from 4.2.2, we have

7.5.21 $$1_{G_J}^G = \sum_{J\subseteq S} \eta_S \qquad \text{for all } J \subseteq \underline{R} .$$

Using the correspondence mentioned in 7.2.30, equations 7.5.20 and 7.5.21 will give

$$\varepsilon^W_{W_{\bar{J}}} = \sum_{L \subseteq J} \omega_L$$

and

$$1^W_{W_J} = \sum_{J \subseteq S \subseteq \underline{R}} \omega_S ,$$

for all $J \subseteq R$, where $\varepsilon_{W_{\bar{J}}}$ is the alternating character of $W_{\bar{J}}$ (note that $\varepsilon_{W_{\bar{J}}}$ corresponds to $St_{G_{\bar{J}}}$ under the correspondence of 7.2.30 (see [C1], Thm. 1)).

<u>Definition 7.5.23</u> If $[\lambda]$ is a diagram of $\lambda \in P$, the <u>conjugate diagram</u> $[\tilde{\lambda}]$ is obtained by interchanging the rows and the columns in $[\lambda]$. $\tilde{\lambda}$ is the partition of n <u>conjugate</u> to λ.

The following lemma is well-known (see for example [G4], §5).

<u>Lemma 7.5.24</u> For each $\lambda \in P$, $1^W_{W(\lambda)}$ and $\varepsilon^W_{W(\tilde{\lambda})}$ have a unique irreducible character of W in common, namely ξ_λ. □

<u>Proposition 7.5.25</u> Suppose that $J \subseteq \underline{R}$ is such that $\lambda(\bar{J}) = \widetilde{\lambda(J)}$. Then, the virtual character ω_J of W is irreducible and equal to $\xi_{\lambda(J)}$.

<u>Proof</u> It is clear from 7.5.22 that, for every $J \subseteq \underline{R}$, $\varepsilon^W_{W_{\bar{J}}}$ and $1^W_{W_J}$ have the character ω_J in common. Now suppose that $J \subseteq \underline{R}$ with $\lambda(\bar{J}) = \widetilde{\lambda(J)}$, then we have $W(\lambda(\bar{J})) = W(\widetilde{\lambda(J)})$, and so $\varepsilon^W_{W_{\bar{J}}} = \varepsilon^W_{W(\widetilde{\lambda(J)})}$. But by 7.5.24, $\varepsilon^W_{W(\widetilde{\lambda(J)})}$ and $1^W_{W(\lambda(J))}$ have a unique irreducible character in common, namely $\xi_{\lambda(J)}$. Therefore $\omega_J = \xi_{\lambda(J)}$. This completes the proof of 7.5.25. □

Example $G = GL(3,q)$.

Every $\chi \in \tilde{H}$ has the form $\chi(a_1,a_2,a_3)$, where, for $i = 1,2,3$, $0 \le a_i < q-1$. The Weyl group $W = S_3 = \langle w_1 = (12), w_2 = (23)\rangle$. We choose the coset representatives of H in N as before. Let $\chi = \chi(a_1,a_2,a_3) \in \tilde{H}$. Then

$$\chi|_{H_1} = 1 \iff \Gamma(x)^{a_1-a_2} = 1 \quad \text{for all} \quad x \in \mathbb{F}_q^\times .$$

$$\iff a_1 \equiv a_2 \pmod{q-1} .$$

Similarly

$$\chi|_{H_2} = 1 \iff a_2 \equiv a_3 \pmod{q-1} .$$

Therefore

$$\text{7.5.26} \qquad P(\chi(a_1,a_2,a_3)) = \begin{cases} \phi & \text{if } a_1 \not\equiv a_2 \not\equiv a_3 \pmod{q-1} \\ \{w_1\} & \text{if } a_1 \equiv a_2 \ ; \ a_2 \not\equiv a_3 \pmod{q-1} \\ \{w_2\} & \text{if } a_1 \not\equiv a_2 \ ; \ a_2 \equiv a_3 \pmod{q-1} \\ \{w_1,w_2\} & \text{if } a_1 \equiv a_2 \equiv a_3 \pmod{q-1} . \end{cases}$$

We discuss the following three cases:

I. $\chi = \chi(a_1,a_2,a_3)$, $a_1 \not\equiv a_2 \not\equiv a_3 \pmod{q-1}$. It is clear that $w\chi \neq \chi$ for all $w \in W$. Hence χ is regular, and so the KG-module $KY_\chi = e_\chi(KG[U])$ is simple KG-module.

By 7.5.26, $P(\chi) = \phi$ and so the RG-lattice $Y_\chi = e_\chi(RG[U])$ is indecomposable. The decomposition matrix D_χ is the 1×1 matrix (1) .

II. $\chi = \chi(a_1,a_2,a_3)$, $a_1 \equiv a_2$, $a_2 \not\equiv a_3 \pmod{q-1}$.

In this case $P(\chi) = \{w_1\}$, and $W_\chi = \langle w_1\rangle$. Let $v = w_2w_1$, then it is clear that

$$(\chi) = \{\chi,\ v\chi,\ v^2\chi\}$$

$$= \{\chi(a_1,a_2,a_3),\ \chi(a_1,a_3,a_2),\ \chi(a_3,a_2,a_1)\}\quad .$$

We note that although $KY_\lambda \cong KY_\mu$ as KG-modules for all $\lambda,\mu \in (\chi)$ (5.0.14(ii)), the RG-lattices $Y_\lambda;\ \lambda \in (\chi)$ are mutually non-isomorpic (in fact $\{Y_\lambda,\ \lambda \in (\chi)\}$ are different R-forms for the same KG-module). However we will discuss the three different cases.

(i) First, consider the KG-module $KY_\chi = e_\chi(KG[U])$. Since $W_\chi = \langle w_1 \rangle$, the R-order E_χ $(= e_\chi E e_\chi)$ has R-basis $\{e_\chi,\ e_\chi A_{(w_1)}\}$ with multiplication relations:

7.5.27 and
$$e_\chi A_{(w_1)} e_\chi = e_\chi A_{(w_1)}\quad ,$$
$$e_\chi A^2_{(w_1)} = q\, e_\chi + (q-1)\, e_\chi A_{(w_1)}\quad .$$

So, putting $a = e_\chi A_{(w_1)}$, we have

$$a^2 - (q-1)a - q = 0 \quad , \qquad \text{hence}$$
$$(a+1)(a-q) = 0 \quad .$$

From the last equation we deduce an orthogonal primitive idempotent decomposition

7.5.28 $\quad e_\chi = e_1 + e_2$

of e_χ in E_χ , where $e_1 = \dfrac{1}{q+1}(a + e_\chi)$, and $e_2 = \dfrac{-1}{q+1}(a - qe_\chi)$.
Therefore, we have

7.5.29 $\quad Y_\chi = e_1(Y_\chi) \oplus e_2(Y_\chi) = X_1 \oplus X_2$,

where $\quad X_i = e_i(Y_\chi)$, $i = 1,2$.

By applying the functor $\otimes_R K$ to equation 7.5.27, we get a similar decomposition for $\tilde{e}_\chi = e_\chi \otimes_R K = |H|^{-1} \sum_{h\in H} \chi(h^{-1})A_{h,K}$. In fact $\tilde{e}_1 = e_1 \otimes K$ and $\tilde{e}_2 = e_2 \otimes_R K$ remain primitive in KE_χ and

$$\tilde{e}_\chi = \tilde{e}_1 + \tilde{e}_2$$

is an orthogonal primitive idempotent decomposition of $\tilde{e}_\chi$ in KE_χ . Consequently

7.5.30 $\qquad KY_\chi = \tilde{e}_1(KY_\chi) \oplus \tilde{e}_2(KY_\chi) = KX_1 \oplus KX_2$,

where KX_1 and KX_2 are simple KG-components of KY_χ of dimensions $q^2 + q + 1$ and $q(q^2 + q + 1)$, respectively.

By reducing equation 7.5.28 mod(π) , we get a primitive orthogonal idempotent decomposition

$$\bar{e}_\chi = \bar{e}_1 + \bar{e}_2$$

of $\bar{e}_\chi$ $(= e_\chi \otimes \mathrm{Id}F)$ in FE_χ . It follows that

$$\begin{aligned} FY_\chi &= \bar{e}_1(FY_\chi) \oplus \bar{e}_2(FY_\chi) \\ &= FY(\chi,\{w_1\}) \oplus FY(\chi,\phi) , \end{aligned}$$

since $P(\chi) = \{w_1\}$. By comparing the dimensions (see [NT2], Thm. 4.11), we have

$$\bar{e}_1(FY_\chi) = FY(\chi,\{w_1\}) \text{ and } \bar{e}_2(FY_\chi) = FY(\chi,\phi) .$$

Therefore, $Y(\chi,\{w_1\}) = e_1(Y_\chi) = X_1$ and $Y(\chi,\phi) = e_2(Y_\chi) = X_2$, and so

$$D_\chi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

(ii) We now consider the system $(KE_{v\chi}, E_{v\chi}, FE_{v\chi})$.

$v\chi = v\chi(a_1,a_2,a_3) = \chi(a_1,a_3,a_2)$, hence $P(v\chi) = \phi$ and $W_{v\chi} = \{1,w_o\}$ (note that $W_{v\chi}$ is not a reflection subgroup). Since $P(v\chi) = \phi$, $FY_{v\chi}$ $(= \bar{e}_{v\chi}(FY))$ is an indecomposable FG-module, hence $Y_{v\chi}$ $(= e_{v\chi}(Y))$ is also indecomposable. Consequently, both $e_{v\chi}$ and $\bar{e}_{v\chi}$ are primitive in $E_{v\chi}$ and $FE_{v\chi}$, respectively.

The K-algebra $KE_{v\chi}$ has K-basis $\{\tilde{e}_{v\chi}, \tilde{e}_{v\chi} A_{(w_o),K}\}$, with multiplication relations given by

7.5.31 and
$$\tilde{e}_{v\chi} A_{(w_o),K}\, \tilde{e}_{v\chi} = \tilde{e}_{v\chi} A_{(w_o),K} ,$$
$$\tilde{e}_{v\chi} A^2_{(w_o),K} = q^3\, \tilde{e}_{v\chi} + q(q-1)\, \tilde{e}_{v\chi} A_{(w_o),K} .$$

From the relation 7.5.31, we deduce an orthogonal primitive idempotent decomposition

$$\tilde{e}_{v\chi} = t_1 + t_2$$

of $\tilde{e}_{v\chi}$ in $KE_{v\chi}$, where $t_1 = \dfrac{1}{q(q+1)} (\tilde{e}_{v\chi} A_{(w_o),K} + q\tilde{e}_{v\chi})$ and $t_2 = \dfrac{-1}{q(q+1)} (\tilde{e}_{v\chi} A_{(w_o),K} - q^2\tilde{e}_{v\chi})$. Hence

$$KY_{v\chi} = t_1(KY_{v\chi}) \oplus t_2(KY_{v\chi}) \cong KX_1 \oplus KX_2 , \quad \text{and so}$$
$$D_{v\chi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

(iii) Consider the system $(KE_{v^2\chi}, E_{v^2\chi}, FE_{v^2\chi})$. $v^2\chi = \chi(a_3,a_1,a_2)$ and so $P(v^2\chi) = \{w_2\}$ (remember that $a_1 \equiv a_2 \pmod{q-1}$) and $W_{v^2\chi} = \{1,w_2\}$. It follows that the R-order $E_{v^2\chi}$ has R-basis $\{e_{v^2\chi}, e_{v^2\chi} A_{(w_2)}\}$ with multiplication relations

$$e_{v^2\chi} A_{(w_2)} e_{v^2\chi} = e_{v^2\chi} A_{(w_2)}$$

7.5.32 and

$$e_{v^2\chi} A^2_{(w_2)} = qe_{v^2\chi} + (q-1)e_{v^2\chi} A_{(w_2)} \quad .$$

This case is similar to case (i) above and so we will have

$$D_{v^2\chi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

<u>Note</u> that $v^2\chi = w_0\chi$ and case (i) and (iii) above verify theorem 7.3.26

III. $\chi = \chi(a_1,a_2,a_3)$, $a_1 \equiv a_2 \equiv a_3 \pmod{q-1}$.

In this case $P(\chi) = \underline{R} = \{w_1,w_2\}$ and so $Y_\chi = Y(\chi,\underline{R}) \underset{R}{\otimes} Y_1$, by 6.0.20, where $Y_1 = RG[B]$. Therefore, we may assume that $\chi = 1$ $(= \chi(0,0,0))$. By 7.2.30, the rows of the decomposition matrix D_1 are indexed by the set of partitions $\{(1^3), (2,1), (3)\}$ of $n = 3$. Since $P(1) = \underline{R}$, the columns of D_1 are indexed by all subsets of $\underline{R}$. However D_1 has the form

$$D_1 = \begin{array}{c} \\ (1^3) \\ (2,1) \\ (3) \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} \phi & \{w_1\} & \{w_2\} & \underline{R} \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix} \end{array} .$$

The following tables give the decomposition matrix D_1 of the system (KE_1, E_1, FE_1) , for $n = 4$ and $n = 5$.

$n = 4$

λ \ J ($d_{\lambda,J}$)	ϕ	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1,w_2\}$	$\{w_1,w_3\}$	$\{w_2,w_3\}$	$\underline{R}$
(4)								1
$(3,1)$					1	1	1	
(2^2)			1			1		
$(2,1^2)$		1	1	1				
1^4	1							

Table 7.5.34: The decomposition matrix D_1 of the system (KE_1, E_1, FE_1), where $n = 4$.

n = 5

$\lambda \backslash J$, $d_{\lambda,J}$	ϕ	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1,w_2\}$	$\{w_1,w_3\}$	$\{w_1,w_4\}$	$\{w_2,w_3\}$	$\{w_2,w_4\}$	$\{w_3,w_4\}$	$\{w_1,w_2,w_3\}$	$\{w_1,w_2,w_4\}$	$\{w_1,w_3,w_4\}$	$\{w_2,w_3,w_4\}$	$\underline{R}$
(5)																1
(4,1)												1	1	1	1	
(3,2)							1		1	1			1	1		
$(3,1^2)$						1	1	1	1	1	1					
$(2^2,1)$			1	1			1	1		1						
$(2,1^3)$		1	1	1	1											
(1^5)	1															

Table 7.5.33: The decomposition matrix D_1 of the system (KE_1,E_1,FE_1) , where n = 5 .

Appendix

Let Λ be a finite dimensional algebra over a field k . In this appendix we prove two lemmas which have been used in various places in the thesis.

Let V be a finitely generated left Λ-module. Suppose that

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_d$$

is a decomposition of V into a direct sum of Λ-modules. For each $i \quad \{1,2,\ldots,d\}$, let $\pi_i : V \to V_i$ be the projection of V onto V_i , and $\mu_i : V_i \to V$ be the inclusion map. Then, putting $e_i = \mu_i\pi_i$, e_i is idempotent in $E(V)$ and $1_{E(V)} = \sum_i e_i$ is a primitive decomposition. We also have $\pi_i\mu_i$ = identity map on V_i .

<u>Lemma 1</u> : $\mathrm{End}_\Lambda(V_i) \cong e_i\ \mathrm{End}_\Lambda(V)e_i$ as k-algebras, for all $i \in \{1,\ldots,d\}$.

<u>Proof</u> Let $\phi : \mathrm{End}_\Lambda(V_i) \to \mathrm{End}_\Lambda(V)$ be the k-map given by

$$\phi(f) := \mu_i\ f\ \pi_i$$

$$\begin{tikzcd} V_i \arrow[d, "f"'] & V \arrow[l, "\pi_i"'] \arrow[d, "\mu_i f \pi_i"] \\ V_i \arrow[r, "\mu_i"'] & V \end{tikzcd}$$

for $f \in E_\Lambda(V_i)$.
ϕ is not an algebra map
(since $\phi(1_{E(V_i)}) \neq 1_{E(V)}$) but we have

$$\begin{aligned} \text{Image of } \phi &= \mu_i\ E_\Lambda(V_i)\pi_i \\ &= \mu_i(\pi_i\mu_i)E_\Lambda(V_i)(\pi_i\mu_i)\pi_i \\ &= (\mu_i\pi_i)\mu_i E\ (V_i)\pi_i(\mu_i\pi_i) \\ &\subseteq e_i\ E_\Lambda(V)e_i \ . \end{aligned}$$

Therefore we may regard ϕ as a map $E_\Lambda(V_i) \to e_iE_\Lambda(V)e_i$, in which case ϕ is obviously k-algebra map. The k-map $\phi' : e_iE_\Lambda(V)e_i \to E_\Lambda(V_i)$

$$\phi'(\xi) := \pi_i\xi\mu_i$$

$$\begin{tikzcd} V_i \arrow[r, "\mu_i"] \arrow[d, "\pi_i\xi\mu_i"'] & V \arrow[d, "\xi"] \\ V_i & V \arrow[l, "\pi_i"] \end{tikzcd}$$

($\xi \in E_\Lambda(V)$) defines an inverse of ϕ . Therefore we have

$$E_\Lambda(V_i) \cong e_iE(V)e_i \quad \text{as k-algebras.} \qquad \square$$

<u>Lemma 2</u> (Triangle Lemma): Let $X_1, X_2, X_3 \in \text{mod } \Lambda$ and let $f \in (X_1,X_2)_\Lambda$, $g \in (X_2,X_3)_\Lambda$. If gf is isomorphism then

$$X_2 = \text{Imf} \oplus \text{Ker } g .$$

<u>Proof</u> Let $x \in X_2$. Put $x_1 = (f(gf)^{-1}g)(x)$ and $x_2 = x - x_1$. It is clear that $x_1 \in \text{Imf}$ and x_2 Ker g , since $g(x_2) = g(x) - (gf(gf)^{-1}g)(x) = g(x) - g(x) = 0$. Hence $x = x_1 + x_2$ and so $X_2 = \text{Imf} + \text{Ker } g$. To finish the proof we must show that $\text{Imf} \cap \text{Ker } g = 0$. Let $y = f(x)$ $(x \in X_1)$ lie in $\text{Imf} \cap \text{Ker } g$. Then $0 = g(y) = (gf)(x)$, but since gf is isomorphism, we have $x = 0$, and hence $y = 0$. Therefore we have

$$X_2 = \text{Imf} \oplus \text{Ker } g$$

and this completes the proof of lemma 2. $\square$

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